

§21.3 SEP. OF VAR'S IN POLAR COORD'S

(1)

LAPLACE EQUATION IN POLAR COORD'S §21.3.1

$$\nabla^2 u = 0$$

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \right] u = 0$$

LET $u = P(\rho) \cdot \Phi(\phi)$

$$P'' \Phi + \frac{1}{\rho} P' \Phi + \frac{1}{\rho^2} P \Phi'' = 0$$

MULTIPLY BY $\rho^2 / (P \Phi)$

$$\rho^2 \frac{P''}{P} + \rho \frac{P'}{P} + \frac{\Phi''}{\Phi} = 0$$

AZIMUTHAL PART

$$\Rightarrow \Phi'' = -n^2 \Phi \quad (\text{WHY } -n^2?)$$

$$\Phi = A e^{in\phi} + B e^{-in\phi}$$

RADIAL PART

$$\Rightarrow \rho^2 P'' + \rho P' = n^2 P$$

Q: HOW DO WE SOLVE THIS?

(DISCUSS USING TABLE 17.1 -- BUT IT'S A DEAD END!) → SEE P. 2

"WHEN ALL YOU HAVE IS A HAMMER, EVERYTHING LOOKS LIKE A NAIL."

ALT. APPROACH:

PUT INTO STURM-LIOUVILLE FORM

(2)

$$\rho^2 P'' + \rho P' = n^2 P$$

$$\rho P'' + P' = \frac{n^2}{\rho} P$$

DOESN'T FOLLOW FORM $\mathcal{D}P = \lambda P$,

SO $\frac{n^2}{\rho} P$ MUST BE PART OF THE OPERATOR:

$$\left[\rho \frac{d^2}{d\rho^2} + \frac{d}{d\rho} - \frac{n^2}{\rho} \right] P = 0$$

BESSEL'S EQ. WITH ZERO EIGENVALUE! $\left(\xi(x) = -\frac{n^2}{x} \right)$

COMPARING WITH TABLE 17.1 —

WE WANT $p(x) = x$

$$q(x) = 0$$

WEIGHTING FACTOR $\rightarrow \rho(x) = 1/x$

~~X~~ ENTRY WITH THESE SPECIFICATIONS. WE NEED EITHER A BETTER REFERENCE, OR A TRICK.

NOTE THAT IN STURM-LIOUVILLE FORM, WE HAVE ONLY $\lambda = 0$ AVAILABLE ANYWAY. IT DOESN'T LOOK PROMISING.

TRICK: USE $\rho = e^t$ TO GET AN ODE WITH CONSTANT COEFFICIENTS.

THIS LEADS TO

$$\frac{dP}{dt} = \frac{dP}{dp} \frac{dp}{dt} = P' p$$

$$\frac{d^2P}{dt^2} = \frac{d}{dt} (pP') = \underbrace{pP' + p^2P''}_{\text{AHA!}}$$

SO WE HAVE

$$\frac{d^2P}{dt^2} = n^2 P$$

$$\Rightarrow P = C e^{nt} + D e^{-nt}$$

$$P = C p^n + D p^{-n}$$

Q: DOES THIS GIVE US AN ORTHONORMAL BASIS?

A: WE COULD USE $n \in \mathbb{R}$ FOR A COMPLETE BUT NON-ORTHOGONAL BASIS SET (POLYNOMIALS IN p OR IN $1/p$).

OR TAKE $n \in \mathbb{Z}$, AND USE THE FOURIER BASIS.

Q: THERE IS ONE LITTLE THING WE MISSED IN THE SOLUTION OF THE $P(\rho)$ EQUATION. WHAT IS IT?

A: THE CASE $n=0$:

$$\rho^2 P'' + \rho P' = 0$$

$$\rho P'' + P' = 0$$

$$\rho \frac{dP'}{d\rho} = -P'$$

$$-\frac{dP'}{P'} = \frac{d\rho}{\rho}$$

$$\ln P' = -\ln \rho + \text{const.}$$

$$P' = D/\rho$$

$$dP = D \frac{d\rho}{\rho}$$

$$P = C + D \ln \rho$$

OR

$$\frac{d^2 P}{dt^2} = 0$$

$$P = C + Dt$$

RECALL $\rho = e^t$, SO

$$P = C + D \ln \rho, \quad n=0$$

WE WOULDN'T GET THE $\ln \rho$ TERM FROM THE SOLN. FOR GENERAL n .

OUR SOLN. IS THEN:

$$U(\rho, \varphi) = C_0 \ln \rho + D_0 + \sum_{n=1}^{\infty} (A_n \cos n\varphi + B_n \sin n\varphi)(C_n \rho^n + D_n \rho^{-n})$$

★ EXERCISES ★

SOLVE FOR THE FOLLOWING B.C.'S:

$$(a) \quad u(a, \varphi) = u_0 \cos \varphi, \quad 0 < a < \infty$$

ON THE DOMAIN $a \leq \rho < \infty \quad 0 \leq \varphi < 2\pi$.

$$(b) \quad u = u_0 e^{-k\rho}, \quad k \in \mathbb{R}^+ \quad \text{ON } \varphi = 0 \text{ AND } \varphi = \pi/2$$

ON THE DOMAIN $\rho > 0, \quad 0 \leq \varphi \leq \pi/2$

NOTE THAT $e^{-x} = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}$

$$(c) \quad u(a, \varphi) = \begin{cases} u_0, & 0 \leq \varphi < \pi \\ 0, & \pi \leq \varphi < 2\pi \end{cases}$$

(i) DOMAIN $\rho \geq a$

(ii) DOMAIN $0 < \rho < a$

(d) CAN YOU FIND A B.C. FOR WHICH OUR GENERAL SOLUTION FAILS? *

$$* \quad u(\rho, 0) = \begin{cases} 1, & \rho \leq a \\ 0, & \rho > a \end{cases} \quad \text{MIGHT DO IT.}$$

WE COULD DO THE SAME PROBLEM IN CARTESIAN COORDINATES... (6)

$$\nabla^2 u = 0$$

DOMAIN: $0 < x < \infty$, $0 < y < \infty$

BOUNDARY CONDITIONS:

$$\left. \begin{array}{l} \text{a. } u(0, y) = u_0 e^{-k_y y} \\ \text{b. } u(x, 0) = u_0 e^{-k_x x} \end{array} \right\} \text{ } k \in \mathbb{R}^+ \text{ IS A CONSTANT.}$$

SEPARATION OF VARIABLES:

$$u = X(x)Y(y)$$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

CONSIDER THESE TWO SOLUTIONS:

I. $X = \sin k_x x$, $Y = e^{-k_y y}$
SATISFIES $u = 0$ ON $y \rightarrow \infty$, AND AT $x = 0$.

II. $X = e^{-k_x x}$, $Y = \sin k_y y$
SATISFIES $u = 0$ ON $x \rightarrow \infty$, AND AT $y = 0$.

SOLUTION I. CAN BE USED TO SATISFY BOUNDARY CONDITION b AT $y = 0$, WITH HOMOGENEOUS DIRICHLET BOUNDARIES ELSEWHERE. SIMILARLY, SOLUTION II. CAN BE USED TO SATISFY B.C. a AT $x = 0$, WITH HOMOGENEOUS DIRICHLET BOUNDARIES ELSEWHERE.

FIRST, WE TURN OUR ATTENTION TO BUILDING UP THE FUNCTION $e^{-k_x x}$ FROM AN INTEGRAL OVER k_x OF THE BASIS FUNCTION $\sin k_x x$. WE KNOW THAT THE FOURIER TRANSFORM OF AN ODD FUNCTION CONTAINS ONLY SINES (NO COSINES), SO OUR PROBLEM REDUCES TO CALCULATING THE FOURIER TRANSFORM OF

$$\text{sgn}(x) e^{-k|x|}$$

$$\mathcal{F}[\text{sgn}(x) e^{-k|x|}] = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ik_x x} e^{-kx} dx$$

$$\int_0^{\infty} e^{-ik_x x} e^{-kx} dx = \left. \frac{-1}{k+ik_x} e^{-(k+ik_x)x} \right|_0^{\infty}$$

$$= \frac{1}{k+ik_x}$$

THEREFORE, FOR $x > 0$,

$$e^{-kx} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x}}{k+ik_x} dk_x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{i \sin(k_x x)}{k+ik_x} dk_x$$

THE INTENDED CONTRIBUTION FROM SOLUTION I IS THUS:

$$u_I(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(k' x)}{k' - ik} e^{-k' y} dk'$$

WE MAY SIMILARLY WRITE

$$u_{II}(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(k' y)}{k' - ik} e^{-k' x} dk'$$

THE SOLUTION IS $u(x, y) = u_I + u_{II}$.

SINCE I KNOW THAT u_I IS REAL, I'LL TAKE THE REAL PART:

$$u_I = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{k' \sin(k' x)}{k'^2 + k^2} e^{-k' y} dk'$$

WE CAN NOW SEE THAT THE INTEGRAND IS EVEN, SO

$$u_I = \frac{2}{\pi} \int_0^{\infty} \frac{k' \sin(k' x)}{k'^2 + k^2} e^{-k' y} dk'$$

$$u_{II} = \frac{2}{\pi} \int_0^{\infty} \frac{k' \sin(k' y)}{k'^2 + k^2} e^{-k' x} dk'$$

SOLUTION TO EXAMPLE (b) IN POLAR COORDINATES...

GOING BACK TO OUR ORIGINAL SEPARATION OF VARIABLES IN POLAR COORDINATES, WE COULD HAVE DONE IT THIS WAY:

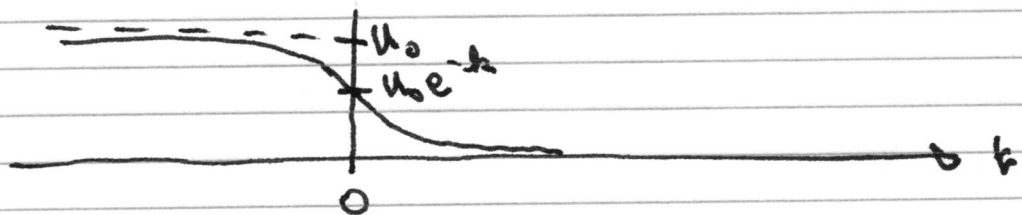
$$\frac{d^2 P}{dt^2} = -\omega^2 P \quad \left| \quad \begin{aligned} \Phi'' &= +\omega^2 \Phi \\ \Phi &= A e^{\omega \varphi} + B e^{-\omega \varphi} \end{aligned} \right.$$

$$P = \frac{1}{\sqrt{2\pi}} e^{i\omega t}, \quad 0 < t < \infty$$

RECALL THAT $\rho = e^t$. OUR BOUNDARY CONDITIONS,

$$u = u_0 e^{-k\rho}, \quad k \in \mathbb{R}^+, \quad \text{AT } \varphi = 0, \pi/2,$$

CAN BE WRITTEN $u = u_0 e^{-Ae^t} \equiv u_B(t)$



IF WE CAN FOURIER TRANSFORM $u_B(t) \rightarrow \tilde{u}_B(\omega)$, @ $\varphi = 0, \pi/2$ THEN WE CAN SUPERIMPOSE SOLUTIONS.

NOTE THAT WE WILL WANT $\Phi(0) = \Phi(\pi/2) = 1$. THAT GIVES US

$$A + B = A e^{\omega\pi/2} + B e^{-\omega\pi/2} = 1, \quad \forall \omega.$$

$$A e^{\omega\pi/2} + (1-A) e^{-\omega\pi/2} = 1$$

$$A = \frac{-e^{-\omega\pi/2}}{e^{\omega\pi/2} - e^{-\omega\pi/2}}$$

$$B = \frac{e^{\omega\pi/2}}{e^{\omega\pi/2} - e^{-\omega\pi/2}}$$

$$\therefore \Phi = \frac{-e^{\omega(\varphi - \pi/2)} + e^{\omega(\pi/2 - \varphi)}}{-e^{-\omega\pi/2} + e^{\omega\pi/2}}$$