

LAPLACE'S EQ. IN (ρ, φ, z)

Q: IS THE ANSWER GOING TO USE THE SAME FUNCTIONS

$$P_n(\rho) = C\rho^n + D\rho^{-n}$$

$$P_0(\rho) = C + D \ln \rho \quad ?$$

A: NO!

TRY IT... THE PDE IS:

$$\nabla^2 u = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

ASSUME $u = P(\rho) \Phi(\varphi) Z(z)$

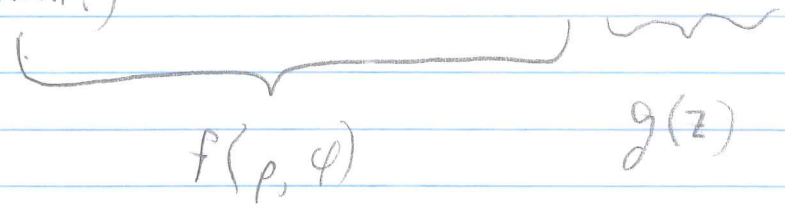
$$P'' \Phi Z + \frac{1}{\rho} P' \Phi Z + \frac{1}{\rho^2} P \Phi'' Z + P \Phi Z'' = 0$$

Q: WHAT'S NEXT?

A: DIVIDE BY u :

$$\frac{P''}{P} + \frac{1}{\rho} \frac{P'}{P} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} + \frac{Z''}{Z} = 0$$

(NOW WHAT?)



$$\frac{Z''}{Z} = k^2 \Rightarrow Z = E e^{-kz} + F e^{kz}$$



AND WE'RE LEFT WITH

$$\frac{P''}{P} + \frac{1}{\rho} \frac{P'}{P} + \frac{1}{\rho^2} \frac{\Phi''}{\Phi} = -k^2$$

MULTIPLY BY ρ^2 :

$$\rho^2 \frac{P''}{P} + \rho \frac{P'}{P} + \rho^2 k^2 + \frac{\Phi''}{\Phi} = 0$$

NB: TERM IN $P(\rho)$ EQ. THAT DID NOT APPEAR IN LAPLACE'S EQUATION FOR JUST ρ AND ϕ !

WE LET

$$\frac{\Phi''}{\Phi} = -m^2, \quad \Phi = C \cos m\phi + D \sin m\phi$$

(AS BEFORE!)

AND THEN WE HAVE

$$\rho^2 P'' + \rho P' + (k^2 \rho^2 - m^2) P = 0$$

Q: WHAT KIND OF EQ. IS THIS?

A: BESSEL'S EQ. OF ORDER m .

$$P = A J_m(k\rho) + B Y_m(k\rho).$$

↙ FIRST KIND ↗ SECOND KIND, BUILT FROM $J_{\pm m}$.

- NOTE THAT $k \rightarrow 0$ DOESN'T RECOVER THE PLANE POLAR SOLUTIONS!
- NOTE THAT Y_m ARE LESS USED BECAUSE $Y_m(x) \rightarrow -\infty$ AS $x \rightarrow 0$.

SOME USEFUL PROPERTIES OF BESSEL FUNCTIONS: (§18.5)

RECURRENCE RELATIONS

$$\left(\frac{d}{dx} [x^\nu J_\nu(x)] \right) = x^\nu J_{\nu-1}(x) \quad (18.92)$$

ORTHOGONALITY

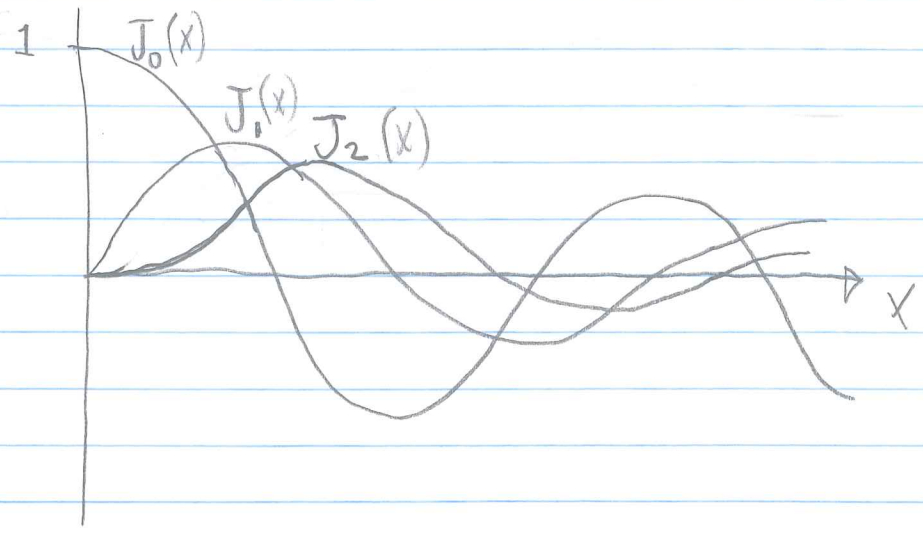
$$\frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x) \quad (18.93)$$

$$\int_a^b x J_\nu(\alpha x) J_\nu(\beta x) dx = 0 \quad \text{FOR } \alpha \neq \beta \quad (18.84)$$

PROVIDED THAT BOTH FUNCTIONS SATISFY HOMOGENEOUS B.C.'S AT a & b. (RECALL OUR SECTION ON STURM-LIOUVILLE OPERATORS).

NORMALIZATION (ASSUMING BC'S FOR ORTHOGONALITY ARE SATISFIED)

$$\int_a^b J_\nu^2(\alpha x) x dx = \frac{1}{2} \left[\left(x^2 - \frac{\nu^2}{\alpha^2} \right) J_\nu^2(\alpha x) + x^2 [J_\nu'(\alpha x)]^2 \right]_a^b \quad (18.89)$$



EXPANSION

A FUNCTION $f(x)$ DEFINED ON $0 \leq x \leq b$, WITH APPROPRIATE BC'S, MAY BE EXPANDED IN TERMS OF $J_\nu(\alpha_n x)$ FOR SOME PARTICULAR ν :

$$f(x) = \sum_{n=0}^{\infty} C_n J_\nu(\alpha_n x) \quad (18.90)$$

DISCUSS ν VS. n .

FINDING COEFFICIENTS C_n :

$$\int_0^b J_\nu(\alpha_n x) f(x) x dx = \int_0^b \left[\sum_{m=0}^{\infty} C_m J_\nu(\alpha_m x) \right] J_\nu(\alpha_n x) x dx$$

ORTHOGONALITY $\Rightarrow = C_n \int_0^b J_\nu^2(\alpha_n x) x dx$

$$= \frac{C_n}{2} \left[\left(x^2 - \frac{\nu^2}{\alpha_n^2} \right) J_\nu^2(\alpha_n x) + x^2 \left[J_\nu'(\alpha_n x) \right]^2 \right]_0^b$$

$$= \frac{C_n}{2} \left[\left(b^2 - \frac{\nu^2}{\alpha_n^2} \right) J_\nu^2(\alpha_n b) + b^2 \left[J_\nu'(\alpha_n b) \right]^2 - \frac{\nu^2}{\alpha_n^2} J_\nu^2(0) \right]$$

DIRICHLET CASE:

IF $f(b) = 0$, THEN WE CHOOSE α_n SO THAT $J_\nu(\alpha_n b) = 0$. THEN:

0 FOR ALL ν
(DISCUSS!)

$$\int_0^b J_\nu(\alpha_n x) f(x) x dx = \frac{C_n}{2} b^2 \left[J_\nu'(\alpha_n b) \right]^2$$

THE RECURSION RELATION (18.93) IS

$$x^{-\nu-1} J_\nu(x) - x^{-\nu} J'_\nu(x) = -x^{-\nu} J_{\nu+1}(x)$$

SO WE HAVE

$$J'_\nu(\alpha_n b) = J_{\nu+1}(\alpha_n b) \quad (\text{RECALL } J_\nu(\alpha_n b) = 0)$$

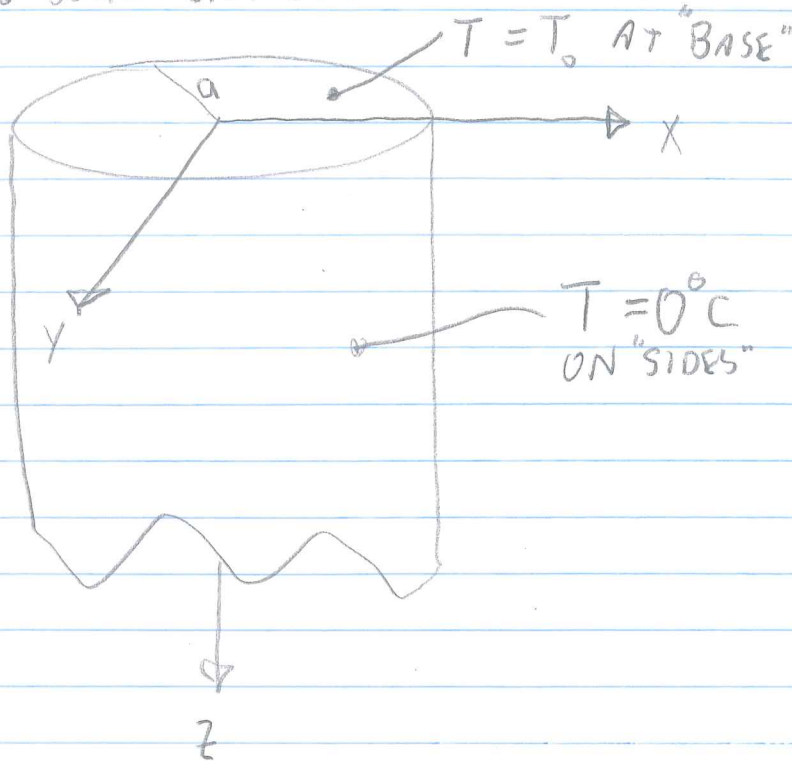
AND THUS

$$C_n = \frac{2}{b^2 J_{\nu+1}^2(\alpha_n b)} \int_0^b f(x) J_\nu(\alpha_n x) x \, dx \quad (18.91)$$

(CASE $f(b) = 0$)

EXAMPLE (P. 729)

SEMI-INFINITE SOLID CYLINDER IN STEADY STATE



WHAT'S THE RIGHT PDE?

$$\frac{\partial T}{\partial t} = \eta \nabla^2 T$$

STEADY STATE $\Rightarrow \partial/\partial t \rightarrow 0$, SO

$$\nabla^2 T = 0$$

(NOTE: $\partial_t T = \eta \nabla^2 T$ IS OFTEN USED FOR "RELAXATION" ALGORITHMS TO COMPUTE SOLUTIONS TO $\nabla^2 T = 0$)

Q: SO WE WANT SOLUTIONS OF WHAT FORM?

A:
$$T(\rho, z) = \sum_{n=0}^{\infty} [A J_0(k_n \rho) + B Y_0(k_n \rho)] [E e^{-k_n z} + F e^{k_n z}]$$

↑
↓
↓

DISCUSS $m=0$
(DISCUSS)
(DISCUSS)

... MAKE THAT

$$T(\rho, z) = \sum_{n=1}^{\infty} A_n J_0(k_n \rho) e^{-k_n z}$$

WHERE k_n IS CHOSEN SO THAT $J_0(k_n a) = 0$.

ROOTS OF J_0 CAN BE LOOKED UP IN ABRAMOWITZ & STEGUN:

$$J_0(x) = 0 \quad \text{FOR } x = 2.40, 5.52, 8.65$$

$$k_1 = 2.40/a, \quad k_2 = 5.52/a, \dots$$

Q: HOW DO WE CHOOSE THE A_n ?

A: BACK TO EQS. 18.90 - 91:

$$T(\rho, 0) = T_0 = \sum_{n=1}^{\infty} A_n J_0(k_n \rho)$$

$$A_n = \frac{2 T_0}{a^2 J_1^2(k_n a)} \int_0^a J_0(k_n \rho) \rho d\rho$$

RECURSION RELATION (18.92) FOR $\nu=1$:

$$\frac{d}{dx} [x J_1(x)] = x J_0(x)$$

THUS

$$J_0(k_n \rho) \rho = \frac{1}{k_n} \frac{d}{d(k_n \rho)} [k_n \rho J_1(k_n \rho)] = \frac{1}{k_n} \frac{d}{d\rho} [\rho J_1(k_n \rho)]$$

SO

$$A_n = \frac{2 T_0}{k_n a J_1(k_n a)}$$

$$T(\rho, z) = \sum_{n=1}^{\infty} \frac{2T_0}{h_n a J_1(h_n a)} J_0(h_n \rho) e^{-h_n z}$$

- WHAT IF WE SET THE WALL TEMPERATURE TO T_1 ?
- WHAT IF $T(\rho, z) = f(\rho)$?
- WHAT IF $T(\rho, \phi, z) = g(\rho, \phi)$?

CHANGE UP THE GEOMETRY OF THE CYLINDER ...

WHAT IF WE HAVE $-\infty < z < \infty$? e^{kz} AND e^{-kz} NO USE!

CLEARLY WE WOULD CHOOSE

$$\frac{z''}{z} = -k^2, \quad z = \frac{1}{\sqrt{2\pi}} e^{ikz} \quad (\text{FOURIER BASIS})$$

WE STILL HAVE

$$\frac{\Phi''}{\Phi} = -m^2, \quad \Phi = C \cos m\phi + D \sin m\phi$$

BUT WHAT HAPPENS TO OUR RADIAL EQUATION?

~~...~~ $\rho^2 P'' + \rho P' - (\rho^2 k^2 + m^2)P = 0$

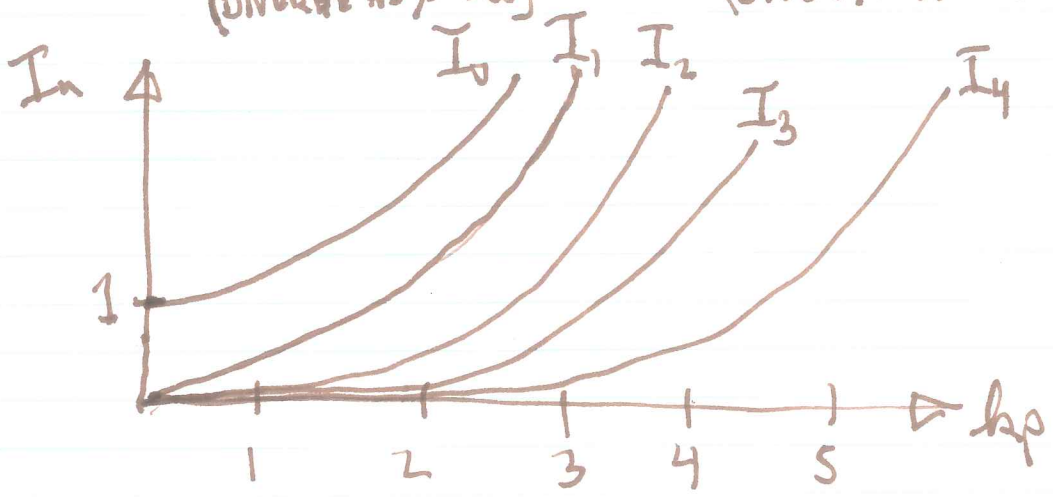
THIS IS A MODIFIED BESSEL EQN.

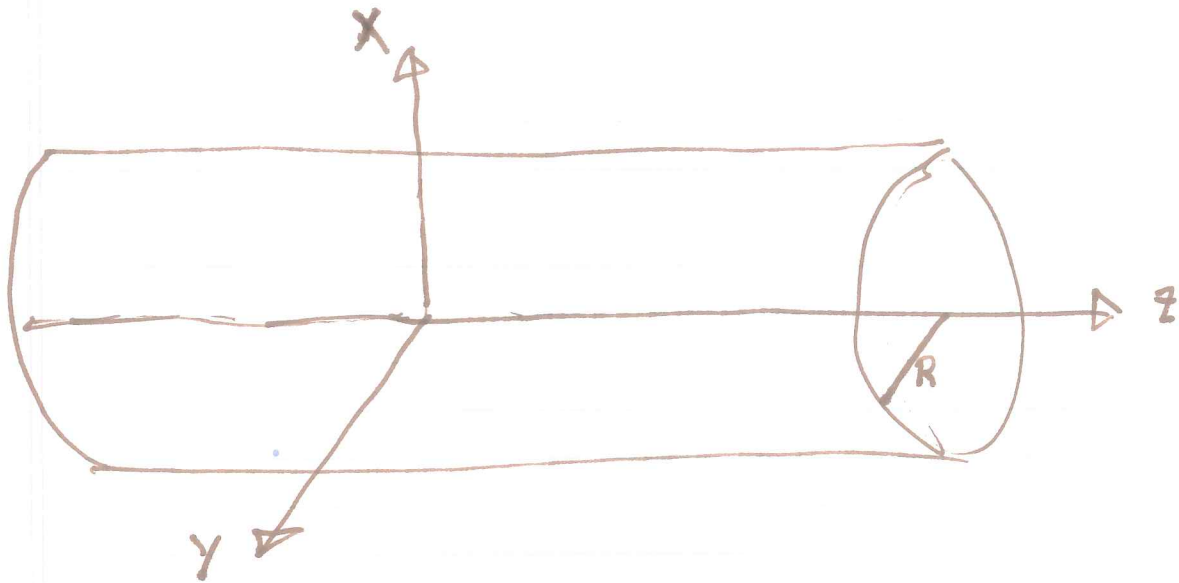
SOLUTIONS ARE:

$$P = A J_m(-ik\rho) + B Y_m(-ik\rho) \\ = a I_m(k\rho) + b K_m(k\rho).$$

MODIFIED BESSEL
1st KIND
(DIVERGE AS $\rho \rightarrow \infty$)

MODIFIED BESSEL
2nd KIND
(DIVERGE AT ORIGIN)





CYLINDER OF RADIUS $\rho = R$, INFINITE LENGTH

SURFACE TEMPERATURE DISTRIBUTION

$$T(R, \phi, z) = \cancel{T(\rho, \phi, z)} = f(\phi, z)$$

FIND INTERNAL EQUILIBRIUM $T(\rho, \phi, z)$.

$$\nabla^2 T = 0$$

SOLUTIONS LOOK LIKE

$$T_{m,h} = (A I_m(h\rho) + B K_m(h\rho)) (C \cos m\phi + D \sin m\phi) \frac{e^{ihz}}{\sqrt{2\pi}}$$

$$-\infty < h_2 < \infty$$

← THE K_m BLOW UP AT THE ORIGIN.

Q: HOW DO WE DEAL WITH THE BOUNDARY CONDITION?

LET'S ASSUME IT IS POSSIBLE TO EXPRESS f IN THE FORM:

$$f(\phi, z) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \left[\tilde{C}_m \cos m\phi + \tilde{D}_m \sin m\phi \right] \frac{e^{ikz}}{\sqrt{2\pi}} dk$$

NOTE THAT $\tilde{C}_m = \tilde{C}_m(k)$, $\tilde{D}_m = \tilde{D}_m(k)$.

HOW DO WE FIND THESE COEFFICIENTS? \rightarrow FOURIER'S TRICK.
(DESPITE THE NAME, THIS "TRICK" WORKS FOR ALL KINDS OF ORTHONORMAL BASIS FUNCTIONS).

CONSIDER

$$\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} f(\phi, z) \cos n\phi d\phi &= \int_{-\infty}^{\infty} \tilde{C}_n(k) \frac{e^{ikz}}{\sqrt{2\pi}} dk \\ &= \mathcal{F}_z^{-1}(\tilde{C}_n(k)). \end{aligned}$$

THUS,

$$\tilde{C}_n(k) = \mathcal{F}_z \left[\frac{1}{\pi} \int_0^{2\pi} f(\phi, z) \cos n\phi d\phi \right]$$

$$\tilde{C}_m = \frac{1}{\pi} \int_0^{2\pi} \tilde{f}(\phi, k) \cos m\phi d\phi$$

SIMILARLY,

$$\tilde{D}_m = \frac{1}{\pi} \int_0^{2\pi} \tilde{f}(\phi, k) \sin m\phi d\phi$$

TO CONSTRUCT THE FULL SOLUTION, WE NEED A WAY OF INSERTING THE RADIAL DEPENDENCE OF EACH TERM IN $f(\varphi, z)$ WHILE LETTING $\underline{P(R) = 1}$ FOR EACH TERM. THAT WAY, WE RECOVER OUR B.C. ...

$$T(\rho, \varphi, z) = \sum_{m=0}^{\infty} \int_{-\infty}^{\infty} \left[\tilde{C}_m \cos m\varphi + \tilde{D}_m \sin m\varphi \right] \times \frac{I_m(k\rho)}{I_m(kR)} \frac{e^{ikz}}{\sqrt{2\pi}} dk$$