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MINOR COMMENT ON STURM-LIOUVILLE FORM:

$$-\frac{1}{p(x)} \left[p(x) u'' + p'(x) u' + q(x) u \right] = \lambda u$$

CAN ALSO BE WRITTEN

$$-\frac{1}{p(x)} \left[\underbrace{\frac{d}{dx} (p(x) u')} + q(x) u \right] = \lambda u.$$

THIS IS THE KEY THING BEHIND SL-FORM.

WE HAVE BEEN IN § 21.3, PERFORMING SOLUTIONS OF

$$\nabla^2 u = 0$$

IN PLANE POLAR, CYLINDRICAL POLAR, AND (TODAY)
SPHERICAL POLAR COORDINATES.

LAPLACE'S Eq. IN SPHERICAL COORDINATES

$$\nabla^2 u = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \star$$

AS USUAL, ASSUME

$$u(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

SO, MULTIPLYING \star BY r^2/u ,

$$\frac{1}{R} \frac{d}{dr} (r^2 R') + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\sin \theta \Theta') + \frac{1}{\Phi \sin^2 \theta} \Phi'' = 0$$

FUNCTION OF r
ONLYFUNCTION OF θ, ϕ ONLY

$$\Rightarrow \frac{1}{R} \frac{d}{dr} (r^2 R') = \lambda,$$

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\sin \theta \Theta') + \frac{1}{\Phi \sin^2 \theta} \Phi'' = -\lambda \quad (21.41)$$

$$r^2 R'' + 2r R' - \lambda R = 0$$

DEFINITELY A STURM-LIOUVILLE EQUATION, BUT NOT IN
TABLE 17.1.

THIS ONE IS AGAIN SUSCEPTIBLE TO THE CHANGE OF VARIABLES:

$$r = e^t$$

$$\frac{dR}{dt} = R' \frac{dr}{dt} = R' r$$

$$\frac{d^2 R}{dt^2} = r \frac{dR'}{dt} + R' r = r^2 R'' + r R'$$

(NOTATION: $R' \equiv \frac{dR}{dr}$)

AND SO OUR ORIGINAL EQ. MAY BE WRITTEN

$$\frac{d^2 R}{dt^2} + \frac{dR}{dt} - \lambda R = 0$$

SINCE WE HAVE CONSTANT COEFF'S, TRY

$$R = e^{\lambda t}$$

$$\Rightarrow \frac{dR}{dt} = \lambda R, \quad \frac{d^2 R}{dt^2} = \lambda^2 R$$

So $\lambda^2 + \lambda - \lambda = 0 \quad \Rightarrow \quad \lambda = \lambda(\lambda + 1)$

$$\lambda = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2} = \frac{-1 \pm \sqrt{1 + 4\lambda(\lambda + 1)}}{2}$$

$$\lambda = \frac{-1 \pm (2\lambda + 1)}{2} = \begin{cases} \lambda \\ -(\lambda + 1) \end{cases}$$

IN OTHER WORDS, FOR A PARTICULAR $\lambda = \lambda(\lambda + 1)$, WE MAY HAVE SOLN'S $e^{\lambda t}$ OR $e^{-(\lambda + 1)t}$.

WE THEREFORE HAVE

$$R = A e^{\lambda t} + B e^{-(\lambda+1)t}$$

$$R(r) = A r^{\lambda} + B r^{-(\lambda+1)}$$

AND 2.41 (MULTIPLIED BY $\sin^2 \theta$, WITH $\lambda \rightarrow \lambda(\lambda+1)$) BECOMES

$$\underbrace{\frac{\sin \theta}{\Theta} \frac{d}{d\theta} (\sin \theta \Theta') + \lambda(\lambda+1) \sin^2 \theta}_{\Theta \text{ ONLY}} + \underbrace{\frac{\Phi''}{\Phi}}_{\Phi \text{ ONLY}} = 0$$

$$\text{LET } \frac{\Phi''}{\Phi} = -m^2, \Rightarrow \Phi = C \cos m\phi + D \sin m\phi$$

AND THEN THE Θ EQUATION IS

CLEARLY A STURM-LIOUVILLE EQ., BUT TABLE 17.1 HAS NO $p(x) = \sin x$.

$$\sin \theta \frac{d}{d\theta} (\sin \theta \Theta') + \lambda(\lambda+1) \sin^2 \theta \Theta = m^2 \Theta \quad (2.44)$$

WE CAN GET RID OF THE TRIG FUNCTIONS WITH THE FOLLOWING CHANGE OF VARIABLES:

$$\text{LET } \mu = \cos \theta, \quad \Theta(\theta) = M(\mu)$$

SO THAT

$$\frac{d}{d\theta} = \frac{d\mu}{d\theta} \frac{d}{d\mu} = -\sin \theta \frac{d}{d\mu} = -\sqrt{1-\mu^2} \frac{d}{d\mu}$$

NOW 2.44 BECOMES:

$$-(1-\mu^2) \frac{d}{d\mu} (-(1-\mu^2) M') + \lambda(\lambda+1) (1-\mu^2) M = m^2 M$$



OR

$$(1-\mu^2) \left[\frac{d}{d\mu} \left((1-\mu^2) M' \right) + \lambda(\lambda+1) M \right] = m^2 M \quad ?!$$

THIS IS A STURM-LIOUVILLE EQUATION ... BUT NO EQUIVALENT IN TABLE 17.1.

WE CAN REARRANGE DIFFERENTLY, LETTING $\lambda(\lambda+1)$ BE THE EIGENVALUE:

$$- \left[\frac{d}{d\mu} \left((1-\mu^2) M' \right) - \frac{m^2}{(1-\mu^2)} M \right] = \lambda(\lambda+1) M$$

AGAIN A STURM-LIOUVILLE EQUATION, NOW RECOGNIZABLE AS THE ASSOCIATED LEGENDRE EQUATION. (§ 18.2)

SOLUTIONS: $M = P_l^m(\mu), Q_l^m(\mu)$

$$\Theta = E P_l^m(\cos \theta) + F Q_l^m(\cos \theta)$$

SO WE HAVE

$$u(r, \theta, \phi) = (A r^l + B r^{-(l+1)}) (C \cos m\phi + D \sin m\phi) \times (E P_l^m(\cos \theta) + F Q_l^m(\cos \theta)) \quad (21.49)$$

WE REQUIRE $m \in \mathbb{Z}$, $l \in \mathbb{Z}$, AND

$$|m| \leq l$$

THESE CONSIDERATIONS
ARISE FROM FORMING
FINITE POLYNOMIAL SOLN'S
OF THE ASSOC. LEG. EQN.

SUMMARY OF THE INTEGER- l LEGENDRE FUNCTIONS: (§18.1)

FOR $m=0$ (THE CASE TREATED, E.G., IN CH.3 OF GRIFFITHS), WE HAVE AZIMUTHAL SYMMETRY. THIS GIVES THE LEGENDRE EQUATION,

$$\frac{d}{dx} \left((1-x^2) M' \right) + l(l+1) M = 0$$

FOR WHICH SOLUTIONS ARE

$$M = a P_l(x) + b Q_l(x)$$

$$P_l(x) \equiv \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2-1)^l \quad (\text{RODRIGUES FORMULA})$$

LEGENDRE POLYNOMIALS

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(5x^2 - 3x)$$

$$P_1(x) = x$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_2(x) = (3x^2 - 1)/2$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

ETC.

THERE EXISTS ANOTHER SET OF SOLUTIONS, LEGENDRE FUNCTIONS OF THE "SECOND KIND":

$$Q_0(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$Q_1(x) = \frac{1}{2} x \ln \left(\frac{1+x}{1-x} \right) - 1$$

∞ AT $x = \pm 1$.

RECURRENCE RELATION:

$$(n+1) P_{n+1} = (2n+1)x P_n - n P_{n-1} \quad (18.27)$$

$$(n+1) Q_{n+1} = (2n+1)x Q_n - n Q_{n-1}$$



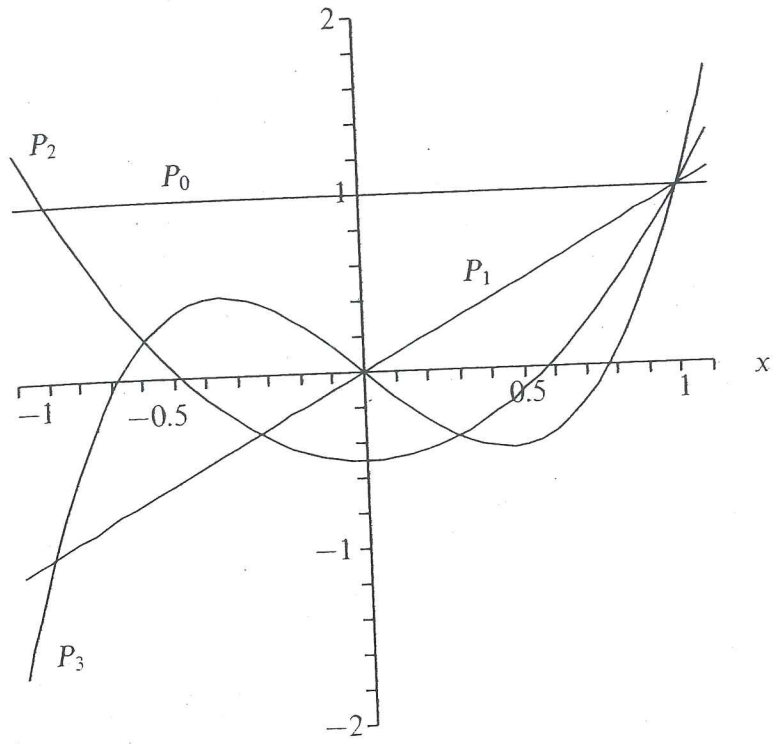


Figure 18.1 The first four Legendre polynomials.

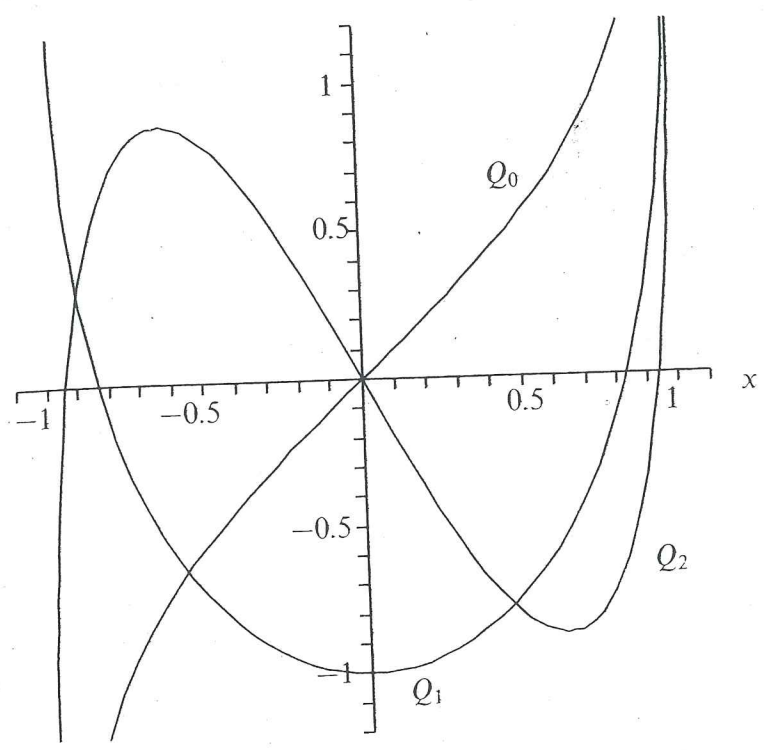


Figure 18.2 The first three Legendre functions of the second kind.

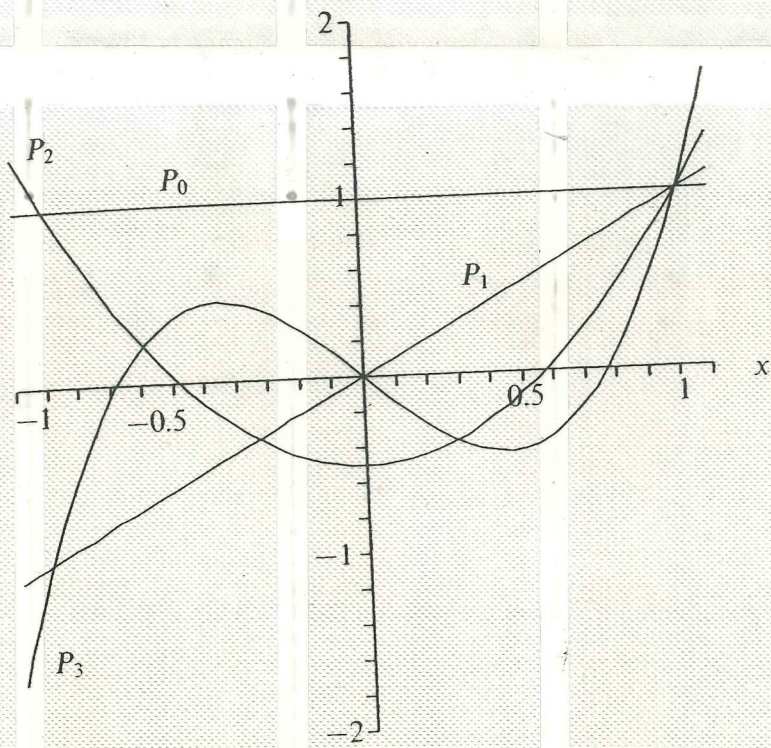


Figure 18.1 The first four Legendre polynomials.

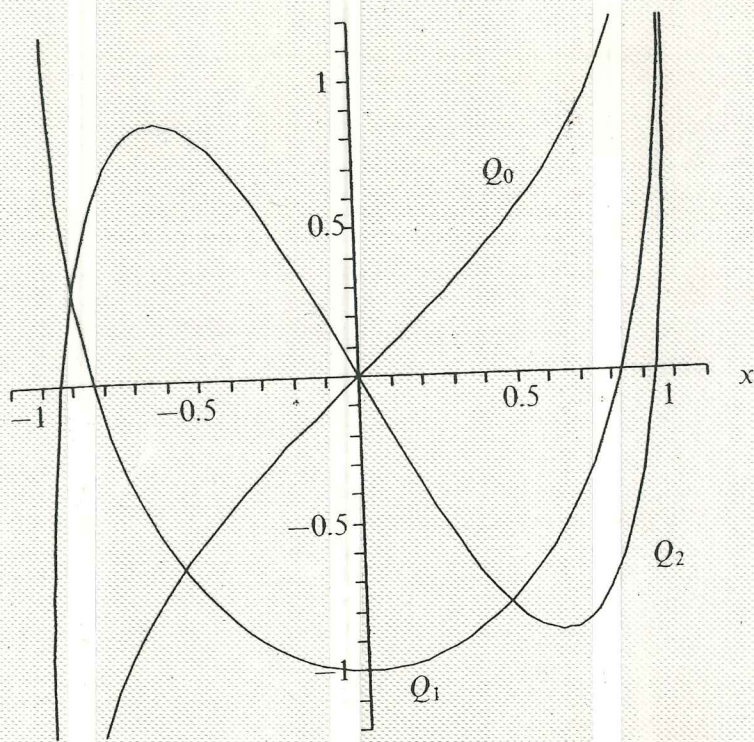


Figure 18.2 The first three Legendre functions of the second kind.

• SINCE IN PHYSICAL APPLICATIONS THE ARGUMENT IS $\cos\theta = x$, THE USUAL DOMAIN OF THE LEGENDRE FUNCTIONS IS $-1 \leq x \leq 1$.

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• NOTE THAT $Q_n(\cos\theta)$ BLOWS UP AT $\theta = 0, \pi$.

• THUS, Q_n 's ARE NOT USED IN CASES WHERE THE Z-AXIS IS IN THE DOMAIN. THAT IS WHY GRIFFITHS IGNORES THEM!

ASSOCIATED LEGENDRE FUNCTIONS

WHEN WE DON'T HAVE AXISYMMETRY, WE MUST DEAL WITH THE FULL EQUATION, ($m \neq 0$)

$$\frac{d}{dm} \left((1-m^2)M' \right) - \frac{m^2}{(1-m^2)} M + l(l+1)M = 0.$$

SOLUTIONS $M = aP_l^m(\mu) + bQ_l^m(\mu)$

WHERE

$$\left. \begin{aligned} P_l^m(x) &= (1-x^2)^{m/2} \left(\frac{d}{dx} \right)^m P_l(x) \\ Q_l^m(x) &= (1-x^2)^{m/2} \left(\frac{d}{dx} \right)^m Q_l(x) \end{aligned} \right\} (18.32)$$

• NOTE THAT m^2 (NOT m) APPEARS IN THE ASSOC. LEGENDRE EQUATION. THIS IMPLIES THAT

$$P_l^m(x) \propto P_l^{-m}(x) \quad (\text{SAME FOR } Q).$$

• SINCE P_l IS A POLYNOMIAL OF ORDER l ,

$$P_l^m(x) = 0 \quad \text{FOR } |m| > l.$$

• ORTHOGONALITY:

$$\int_{-1}^1 P_l^m(x) P_h^m(x) dx = 0 \quad \text{FOR } l \neq h$$

• NORMALIZATION:

$$\int_{-1}^1 [P_l^m(x)]^2 dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

• P_l^m 's ARE A BASIS FOR ANY m !

SPHERICAL HARMONICS

$$Y_l^m(\theta, \phi) \equiv (-1)^m \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos\theta) e^{im\phi}$$

AND $Y_l^{-m} = (-1)^m Y_l^{m*}$

- THESE ARE PROPERLY NORMALIZED!
- FIRST FEW Y_l^m 'S ARE TABULATED IN § 18.45.

$$\int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi Y_l^{m*} Y_{l'}^{m'} = \delta_{ll'} \delta_{mm'}$$

OF COURSE, $d(\cos\theta) = -\sin\theta d\theta$

SO $\int_{-1}^1 d(\cos\theta) = -\int_{\pi}^0 \sin\theta d\theta = \int_0^{\pi} \sin\theta d\theta$

SINCE $d\Omega = \sin\theta d\theta d\phi$, WE HAVE

$$\int Y_l^{m*} Y_{l'}^m d\Omega = \delta_{ll'} \delta_{mm'}$$

OUR SOLUTION TO $\nabla^2 u = 0$ MAY NOW BE EXPRESSED COMPACTLY:

$$u(\rho, \theta, \phi) = \sum_{l,m} (A_{lm} r^l + B_{lm} r^{-(l+1)}) Y_{lm}(\theta, \phi)$$

↑ ↑
DISCUSS NOT JUST A_l, B_l .

$l = 0, 1, 2, \dots$

$m = -l, -l+1, \dots, l$