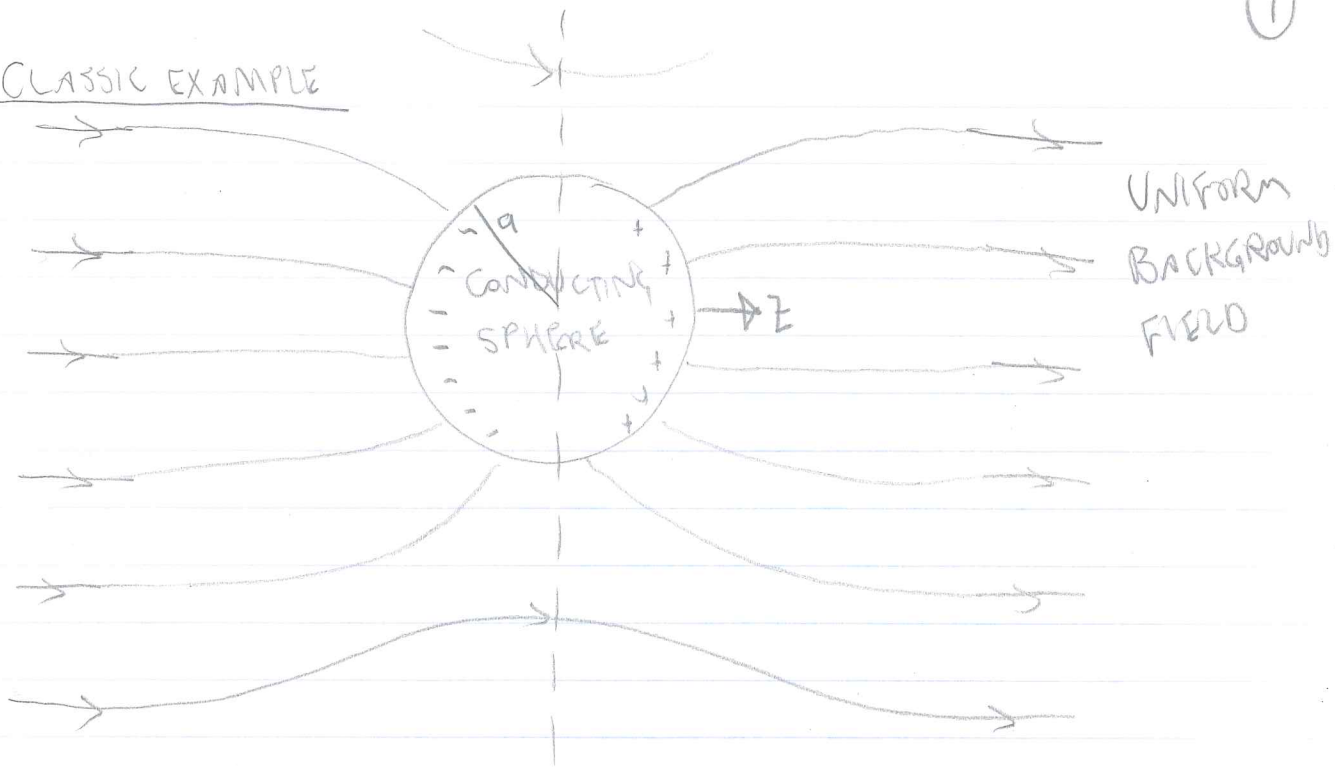


CLASSIC EXAMPLE



- UNIFORM BACKGROUND:

$$\vec{E} = E \hat{z} \Rightarrow \Phi = -Ez = -Er \cos \theta, \quad r \gg a$$

- BY SYMMETRY, THE SPHERE SHOULD HAVE $\Phi = 0$

$$(\Phi(a, \theta, \phi) = 0)$$

- AXIAL SYMMETRY $\Rightarrow m = 0$.

- SOLUTION HAS FORM

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)$$

$P_l^0 = P_l$

SINCE $\Phi(a, \theta, \varphi) = 0$ AND $\langle P_l | P_{l'} \rangle = 0$ FOR $l \neq l'$,

WE REQUIRE THAT

$$\left(A_l r^l + B_l r^{-(l+1)} \right)_{r=a} = 0 \quad \forall l$$

$$\Rightarrow A_l = -B_l a^{-(2l+1)} \quad *$$

THUS A_l CAN ONLY BE ZERO IF B_l IS ZERO.

LOOK AT THE LIMIT $r \rightarrow \infty$:

$$\Phi(r, \theta, \varphi) \rightarrow -Er \cos \theta$$

$$\Rightarrow A_l = 0 \text{ FOR } l = 2, 3, 4, \dots, \text{ AND}$$

$$A_l = 0 \text{ FOR } l = 0$$

$$\Rightarrow B_l = 0 \text{ FOR } l = 0, 2, 3, 4, \dots$$

WE ARE LEFT WITH $l = 1$:

$$\Phi = (A_1 r + B_1 r^{-2}) P_1(\cos \theta) = (A_1 r + B_1 r^{-2}) \cos \theta$$

$$A_1 = E$$

BY *, $B_1 = -E a^3$

$$\therefore \Phi = E \left(r - a^3 r^{-2} \right) \cos \theta$$

UNIFORM
FIELD

ELECTRIC
DIPOLE

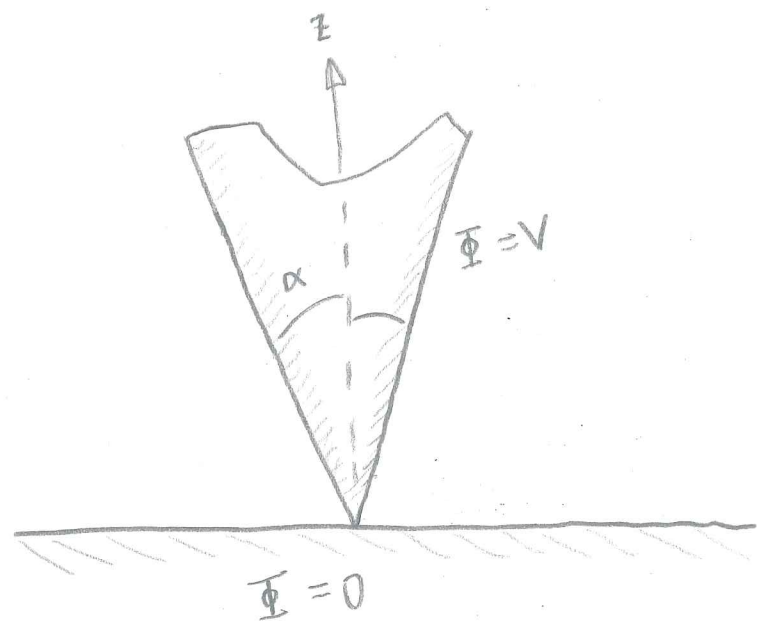
EXAMPLE

A scanning tunneling microscope (STM) tip, modeled as a perfectly conducting cone $\theta \leq \alpha$ in spherical coordinates, is in contact with the surface of a perfectly conducting object occupying the half-space $\theta \geq \pi/2$. The STM tip is held at potential $\Phi = V$, while $\Phi = 0$ within the object. The infinitesimal point of contact (the origin) has resistance R .

- A. Find the magnetic field, $\mathbf{B}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.
- B. Find the electric field, $\mathbf{E}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.
- C. Calculate (or deduce by physical insight) the value of

$$\oint_D \mathbf{S} \cdot d\mathbf{a},$$

where \mathbf{S} is the Poynting flux, and the domain D is the sphere $r = b$. Interpret your result physically.



(B)

(4)

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) \left[C_l P_l(\cos \theta) + D_l Q_l(\cos \theta) \right]$$

↑
INCLUDE Q_l SINCE Z-
AXIS INACCESSIBLE.

THE MAIN CLUE IS THE BOUNDARY CONDITION, WHICH HAS Φ INDEPENDENT OF r FOR $\theta = \alpha$ AND $\theta = \pi/2$. WE SEEM TO HAVE ONLY $l=0$ TO WORK WITH.

$$\begin{aligned} \Phi &= C_0 P_0(\cos \theta) + D_0 Q_0(\cos \theta) \\ &= C_0 + D_0 \frac{1}{2} \ln \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right) \end{aligned}$$

BC #1

AT $\theta = \pi/2$, WE HAVE $\cos \theta = 0$ AND

$$\Phi = 0 = C_0 + \frac{D_0}{2} \ln 1 = C_0 + 0$$

$$\Rightarrow C_0 = 0$$

BC #2

AT $\theta = \alpha$,

$$\Phi = V = \frac{D_0}{2} \ln \left(\frac{1 + \cos \alpha}{1 - \cos \alpha} \right)$$

$$D_0 = \frac{2V}{\ln(1 + \cos \alpha) - \ln(1 - \cos \alpha)}$$

$$\therefore \boxed{\Phi = V \frac{\ln \left(\frac{1 + \cos \theta}{1 - \cos \theta} \right)}{\ln \left(\frac{1 + \cos \alpha}{1 - \cos \alpha} \right)}}$$

IT CAN BE SHOWN THAT THIS MATCHES THE ALTERNATE SOLN. ON THE NEXT PAGE.

"WHEN ALL YOU HAVE IS A HAMMER, EVERYTHING LOOKS LIKE A NAIL."

Problem 8. A scanning tunneling microscope (STM) tip, modeled as a perfectly conducting cone $\theta \leq \alpha$ in spherical coordinates, is in contact with the surface of a perfectly conducting object occupying the half-space $\theta \geq \pi/2$. The STM tip is held at potential $\Phi = V$, while $\Phi = 0$ within the object. The infinitesimal point of contact (the origin) has resistance R .

A. Find the magnetic field, $\mathbf{B}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

The current is $I = V/R$. Taking advantage of the axial symmetry,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

$$\mathbf{B} = -\frac{\mu_0 V \hat{\phi}}{2\pi R \rho},$$

where ρ is the cylindrical radial coordinate.

B. Find the electric field, $\mathbf{E}(\mathbf{r})$, for $\alpha \leq \theta \leq \pi/2$.

The problem is azimuthally symmetric, so $\nabla^2 \Phi(r, \theta) = 0$. Another symmetry: if $r \rightarrow kr$, the boundary conditions do not change. So the equipotentials must be radial, and Φ is a function of θ only. Consequently,

$$0 = \frac{d}{d\theta} \left(\sin \theta \frac{d\Phi}{d\theta} \right).$$

Solving by direct integration and applying the boundary conditions,

$$\Phi(\theta) = C \ln[\tan(\theta/2)] + D = V \frac{\ln[\tan(\theta/2)]}{\ln[\tan(\alpha/2)]}.$$

$$\mathbf{E} = -\nabla \Phi = \frac{-V \hat{\theta}}{2r \ln[\tan(\alpha/2)]} [\cot(\theta/2) + \tan(\theta/2)].$$

C. Calculate (or deduce by physical insight) the value of

$$\oint_D \mathbf{S} \cdot d\mathbf{a},$$

where \mathbf{S} is the Poynting flux, and the domain D is the sphere $r = b$. Interpret your result physically.

Because of Poynting's theorem, $\oint_S \mathbf{S} \cdot d\mathbf{a} = -V^2/R$ for any surface S enclosing the origin. In other words, the fields must be consistent with the dissipation of power in the resistor placed at the origin.

HARDER QUESTION:

SUPPOSE WE ARE SOLVING $\nabla^2 \Phi = 0$ SUBJECT TO B.C.'s

$$\Phi(a, \theta, \varphi) = f(\theta, \varphi),$$

$$\Phi(b, \theta, \varphi) = g(\theta, \varphi),$$

$$b > a, \quad f \neq g$$

Q: HOW WOULD YOU APPROACH THIS?

HINT: BREAK INTO TWO PIECES:

$$\Phi = \Phi_a + \Phi_b, \text{ WITH B.C.'s!}$$

$$\Phi_a(a, \theta, \varphi) = f(\theta, \varphi) \quad \Phi_b(a, \theta, \varphi) = 0$$

$$\Phi_a(b, \theta, \varphi) = 0 \quad \Phi_b(b, \theta, \varphi) = g(\theta, \varphi)$$

Q: HOW DO WE SATISFY THE HOMOGENEOUS DIRICHLET B.C.'s THEN ON Φ_a AT $r=b$ (AND Φ_b AT $r=a$)?

A: $(A_{lm}^a b^l + B_{lm}^a b^{-(l+1)}) = 0.$