

CH. 21 SO FAR

WHAT WE'VE DONE

§ 21.1 SEPARATION OF VARIABLES TECHNIQUE

§ 21.2 SUPERPOSITION OF SEPARATED SOLUTIONS

§ 21.3 SEPARATION IN (PLANE, CYLINDRICAL OR SPHERICAL) POLAR COORDINATES

WHAT I'M SKIPPING (PLEASE READ!)

§ 21.3.2 — HELMHOLTZ EQ. } POSSIBLY HELPFUL FOR HOMEWORK

§ 21.3.3 — SOLUTION BY EXPANSION

§ 21.3.4 — SEPARATION OF VAR'S FOR INHOMOGENEOUS EQ'S

TODAY — MOVING BEYOND SEP. OF VARIABLES

§ 21.4 — INTEGRAL TRANSFORM METHODS
(AND CONNECTION TO EIGENVALUE PROBLEMS)

SOME KEY CONCEPTS!

- SEP. OF VAR'S PROCEDURE
- SOLUTION OF RESULTING ODE'S
 - $x = e^t$ TRICK FOR TRANSFORMING TO AN ODE WITH CONSTANT COEFF'S
 - STURM-LIOUVILLE THEORY
- EXPANSION OF SOLUTIONS IN TERMS OF ORTHONORMAL FUNCTIONS

★ MENTION $\mathcal{F}_{xyz}(f(x,y,z)) = \mathcal{F}_x[\mathcal{F}_y\{\mathcal{F}_z(f)\}]$

§ 21.4

①

FOURIER TRANSFORM APPROACH TO PDE'S

STRATEGY: FOURIER OR LAPLACE TRANSFORM TO REMOVE DERIVATIVES FROM THE EQUATION.

THIS CAN BE DONE WITH FOURIER OR LAPLACE TRANSFORMS, BUT I'LL ONLY DISCUSS FOURIER.

KEY REQUIREMENT: COEFFICIENTS MUST BE CONSTANT WITH RESPECT TO THE TRANSFORM VARIABLE.

Q: WHICH VARIABLE(S) SHOULD I TRANSFORM AWAY?

A: RHB CALLS THIS A "MATTER OF EXPERIENCE," WHICH IS QUITE LAZY ON THEIR PART.

YOU'LL GET THE NECESSARY "EXPERIENCE" TODAY!

EXAMPLES

LAPLACE'S EQUATION: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

① • FT WRT X.

DIFFUSION EQUATION: $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$

② • FT WRT X.

③ • FT WRT t.

WAVE EQUATION: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

④ • FT WRT X

⑤ • FT WRT t

ADVECTION EQUATION: $\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$

⑥ • FT WRT X

⑦ • FT WRT t

→ PICK ONE THAT I'LL DO FIRST

→ GET STUDENTS WORKING ON THE REST. THEY SHOULD GET

(a) FOURIER TRANSFORMED EQ. (ODE!)

(b) SOLUTION TO THE ODE

(c) SOLUTION TO THE PDE

(d) WHAT ARE THE APPROPRIATE ^{OR NATURAL} B.C.'S FOR THIS SOLUTION?

EXAMPLE 6:

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$$

FT WRT X:

(a) $\frac{\partial}{\partial t} \tilde{u}(k, t) = -ikv \tilde{u}(k, t)$

(b) $\Rightarrow \tilde{u}(k, t) = \tilde{u}(k, 0) e^{-ikvt}$

(c) $u(x, t) = \mathcal{F}_x^{-1} [\tilde{u}(k, 0) e^{-ikvt}]$

USING THE FOURIER "TRANSLATION" RULE,

$u(x, t) = u(x-vt, 0)$

(d) THIS SOLN. IS IN TERMS OF INITIAL CONDITIONS (t=0),

$u(x, 0)$

DIFFUSION EQUATION

FT WRT x:

$$\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial}{\partial t} \tilde{u}(k, t) = -\eta k^2 \tilde{u}(k, t)$$

$$\tilde{u}(k, t) = \tilde{u}(k, 0) e^{-\eta k^2 t}$$

$$\Rightarrow u(x, t) = \mathcal{F}_x^{-1} [\tilde{u}(k, 0) e^{-\eta k^2 t}]$$

DISCUSS

FT WRT t:

$$\frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2}$$

RECALL
 $\sqrt{i} = \frac{1}{\sqrt{2}}(1+i)$

$$\frac{i\omega}{\eta} \tilde{u}(x, \omega) = \frac{\partial^2}{\partial x^2} \tilde{u}(x, \omega)$$

$$\tilde{u}(x, \omega) = \tilde{u}(0, \omega) e^{\pm \sqrt{\frac{i\omega}{\eta}} x}$$

$$\tilde{u}(x, \omega) = \tilde{u}(0, \omega) e^{-\sqrt{\frac{i\omega}{2\eta}}(1+i)x}$$

(PHYSICAL REASONS)

FT WRT t & x:

$$i\omega \tilde{u}(k, \omega) = -\eta k^2 \tilde{u}(k, \omega)$$

$$i\omega = -\eta k^2$$

$$u(x, t) = e^{i\omega t} e^{ikx}$$

NOTE THAT WE COULD HAVE DONE THE SAME BY ASSUMING THIS FORM OF SOLN.

④ WAVE EQUATION

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

FT WRT x :

$$(a) \quad \frac{\partial^2 \tilde{u}(k, t)}{\partial t^2} = -c^2 k^2 \tilde{u}(k, t)$$

$$(b) \quad \tilde{u}(k, t) = \tilde{f}(k) e^{i c k t} + \tilde{g}(k) e^{-i c k t} \quad (*!)$$

$$\text{B.C.'s: } \tilde{u}(k, 0) = \tilde{f}(k) + \tilde{g}(k) \equiv \tilde{u}_0$$

$$\dot{\tilde{u}}(k, 0) = i c k (\tilde{f}(k) - \tilde{g}(k)) \equiv \dot{\tilde{u}}_0$$

SOLVING FOR \tilde{f} & \tilde{g} :

$$\dot{\tilde{u}}_0 = i c k (\tilde{u}_0 - \tilde{g}) - \tilde{g}$$

$$\tilde{g} = \frac{1}{2} \left(+ \frac{i}{c k} \dot{\tilde{u}}_0 + \tilde{u}_0 \right)$$

$$\tilde{f} = \tilde{u}_0 - \tilde{g}$$

$$\tilde{f} = \frac{1}{2} \left(- \frac{i}{c k} \dot{\tilde{u}}_0 + \tilde{u}_0 \right)$$

so

$$\tilde{u}(k, t) = \frac{1}{2} \left[\tilde{u}(k, 0) - \frac{i}{c k} \dot{\tilde{u}}_0(k, 0) \right] e^{i c k t}$$

$$+ \frac{1}{2} \left[\tilde{u}(k, 0) + \frac{i}{c k} \dot{\tilde{u}}_0(k, 0) \right] e^{-i c k t}$$

$$\tilde{u}(k, t) = \tilde{u}(k, 0) \cos(c k t) + \frac{1}{c k} \dot{\tilde{u}}_0(k, 0) \sin(c k t)$$

(*!) PHYSICALLY, $f(x)$ AND $g(x)$ ARE INITIAL CONDITIONS FOR THE LEFT- AND RIGHT-GOING WAVES.

(c)

Now, note that $\mathcal{F}_x^{-1}(\cos ckt) = \sqrt{\frac{\pi}{2}} (\delta(x-ct) + \delta(x+ct))$

$\mathcal{F}_x^{-1}(\sin ckt) = i\sqrt{\frac{\pi}{2}} (\delta(x-ct) - \delta(x+ct))$

AND RECALL THAT $\mathcal{F}(f * g) = \sqrt{2\pi} fg$

$$u(x,t) = u(x,0) * \frac{1}{2}(\delta(x-ct) + \delta(x+ct)) + \mathcal{F}_x^{-1}\left(\frac{i}{2ck} \dot{u}(k,0)\right) * (\delta(x-ct) - \delta(x+ct))$$

$\mathcal{F}\left[\int f(t) dt\right] = \frac{1}{i\omega} \mathcal{F} + c\delta(\omega)$

$$= \frac{1}{2} [u(x-ct,0) + u(x+ct,0)] - \frac{1}{2c} \left[\int_a^x \dot{u}(x',0) dx' \right] * [\delta(x-ct) - \delta(x+ct)]$$

ARBITRARY CONSTANT OF INTEGRATION

2nd TERM IS: (CAREFUL!)

$$- \frac{1}{2c} \left[\int_{-\infty}^{\infty} \left(\int_a^z \dot{u}(x',0) dx' \right) (\delta(x-ct-z) - \delta(x+ct-z)) dz \right]$$

$$= - \frac{1}{2c} \left[\int_a^{x-ct} \dot{u}(x',0) dx' - \int_a^{x+ct} \dot{u}(x',0) dx' \right]$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} \dot{u}(x',0) dx'$$

$$u(x,t) = \frac{1}{2} [u(x-ct,0) + u(x+ct,0)] - \frac{1}{2c} \int_{x-ct}^{x+ct} \dot{u}(x',0) dx'$$

cf. CH. 20

(d) THIS IS A SOLN. TO THE INITIAL VALUE PROBLEM. (CAUCHY)

THE QUESTION WAS:

Q: HOW DO YOU KNOW WHICH ^{INDEPENDENT} VARIABLE(S) TO TRANSFORM?

A: TRANSFORM THE VARIABLE(S) ALONG THE BOUNDARY.

Q: GIVEN $\nabla^2 u = 0$, $u(x, y, 0) = f(x, y)$,
WHAT SHOULD I DO?

A: FT WRT x, y .

CHALLENGE: SOLVE LAST WEEK'S PROBLEM 1
(MAGNETIC FIELD EXTRAPOLATION) USING THE ABOVE APPROACH.

i.e. $\nabla^2 \Phi = 0$, $\Phi \rightarrow 0$ AS $z \rightarrow \infty$

GIVEN $B_{z0}(x, y) \equiv - \left. \frac{\partial \Phi}{\partial z} \right|_{z=0}$



RECALL HW 6, #1:

$$\nabla^2 \Phi = 0, \quad \Phi \rightarrow 0 \text{ AS } z \rightarrow \infty$$

$$\text{GIVEN } B_{z0}(x,y) = - \left. \frac{\partial \Phi}{\partial z} \right|_{z=0}$$

FOURIER TRANSFORM IN THE COORDINATES PARALLEL TO THE BOUNDARY PLANE: $(x, y) \rightarrow (k_x, k_y)$

$$\left(-k_x^2 - k_y^2 + \frac{\partial^2}{\partial z^2} \right) \tilde{\Phi}(k_x, k_y, z) = 0$$

$$\frac{\partial^2}{\partial z^2} \tilde{\Phi} = +k^2 \tilde{\Phi} \quad (k^2 \equiv k_x^2 + k_y^2)$$

$$\Rightarrow \tilde{\Phi}(k_x, k_y, z) = A e^{-kz}$$

$$\Rightarrow A = \tilde{\Phi}(k_x, k_y, 0)$$

$$\therefore \tilde{\Phi}(k_x, k_y, z) = \tilde{\Phi}(k_x, k_y, 0) e^{-kz}$$

Q: oops! WRONG B.C. WHAT NOW?

$$A: \mathcal{F}_{xy} \left[-\left. \frac{\partial}{\partial z} \Phi(x,y,z) \right|_{z=0} \right] = \mathcal{F}_{xy} [B_{z0}(x,y)] = \tilde{B}_{z0}(x,y)$$

$$\Rightarrow -\left. \frac{\partial}{\partial z} \tilde{\Phi}(k_x, k_y, z) \right|_{z=0} = +k \tilde{\Phi}(k_x, k_y, 0) = \tilde{B}_{z0}(x,y)$$

$$\Rightarrow \tilde{\Phi}(k_x, k_y, 0) = \frac{\tilde{B}_{z0}}{k}$$

$$\therefore \tilde{\Phi}(k_x, k_y, z) = \frac{\tilde{\Phi}(k_x, k_y, 0) e^{-kz}}{k}$$

SAME AS POSTED HW SOLN.