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EIGENVALUES, EIGENFUNCTIONS & PROPAGATORS (NOT IN TEXT!)

SUPPOSE $\frac{\partial}{\partial t} \varphi(\vec{r}, t) = \mathcal{D} \varphi(\vec{r}, t)$.

↑ DIFFERENTIAL OPERATOR WRT \vec{r} ONLY

IF \mathcal{D} IS A HERMITIAN OPERATOR WITH

$$\mathcal{D} \psi_n(\vec{r}) = \lambda_n \psi_n(\vec{r})$$

THEN CONSIDER AN INITIAL VALUE PROBLEM (THINK: XENON!)

$$\varphi_n(\vec{r}, 0) = \psi_n(\vec{r}),$$

ASSUME A SOLUTION

$$\varphi_n(\vec{r}, t) = \psi_n(\vec{r}) f(t),$$

$$\Rightarrow \mathcal{D} \varphi_n = \frac{\partial}{\partial t} \varphi_n$$

$$\cancel{\lambda_n \psi_n} f = \cancel{\psi_n} \dot{f}$$

$$f = e^{\lambda_n t}$$

$$\Rightarrow \varphi_n(\vec{r}, t) = \psi_n(\vec{r}) e^{\lambda_n t}$$

IF WE HAVE A MORE GENERAL INITIAL CONDITION,

$$\varphi(\vec{r}, 0) = \sum_n c_n \psi_n(\vec{r}),$$

THEN

$$\begin{aligned} \varphi(\vec{r}, t) &= \sum_n e^{\lambda_n t} c_n \psi_n(\vec{r}) = \sum_n e^{\mathcal{D}t} c_n \psi_n(\vec{r}) \\ &= e^{\mathcal{D}t} \sum_n c_n \psi_n(\vec{r}) \end{aligned}$$

$$\boxed{\varphi(\vec{r}, t) = e^{\mathcal{D}t} \varphi(\vec{r}, 0)}$$

EXAMPLES

FIND THE PROPAGATOR FOR THE SCHRÖDINGER EQUATION,

$$H\psi = i\hbar \frac{\partial}{\partial t} \psi.$$

PUT IN FORM

$$\frac{\partial}{\partial t} \psi = \mathcal{D} \psi = \frac{i}{\hbar} H \psi$$

PROPAGATOR IS THEN $e^{\mathcal{D}t} = e^{-iHt/\hbar}$

(FAMILIAR RESULT)

FIND THE PROPAGATOR FOR THE ADVECTION EQUATION,

$$\frac{\partial u}{\partial t} = -v \frac{\partial u}{\partial x}$$

$$e^{\mathcal{D}t} = e^{-vt \partial_x}$$

CHECK THAT THIS IS THE CORRECT SOLN TO THE INITIAL VALUE PROBLEM:

$$u(x,t) = e^{-vt \partial_x} u(x,0)$$

$$\frac{\partial}{\partial t} u(x,t) = -v \partial_x e^{-vt \partial_x} u(x,0)$$

$$= -v \partial_x u(x,t) \quad \checkmark$$

COULD THIS BE DONE FOR THE WAVE EQUATION?

THE ANSWER IS YES.

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

LET $\varphi = \psi$

$$\frac{\partial \varphi}{\partial t} = \frac{c^2 \partial^2 \psi}{\partial x^2}$$

$$\frac{\partial}{\partial t} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \begin{bmatrix} \varphi \\ c^2 \partial^2 \psi / \partial x^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \psi \\ \varphi \end{bmatrix}$$

NOW TO SOLVE THE EIGENVALUE PROBLEM:

$$\begin{bmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \lambda \begin{bmatrix} \psi \\ \varphi \end{bmatrix}$$

$$\varphi = \lambda \psi$$

$$c^2 \psi'' = \lambda \varphi = \lambda^2 \psi$$

$$\psi = e^{ikx}, \quad k = \frac{\lambda}{c}$$

A BETTER NAME FOR THE EIGENVALUE: $\lambda = \omega!$

IS THE PROPAGATOR GOOD FOR ANYTHING? CONSIDER...

$$\begin{aligned} \phi(\vec{r}, dt) &= e^{\mathcal{L} dt} \phi(\vec{r}, 0) \\ &= (1 + \mathcal{L} dt) \phi(\vec{r}, 0) \\ &= \phi(\vec{r}, 0) + dt \mathcal{L} \phi(\vec{r}, 0) \end{aligned}$$

SIMILARLY,

$$\begin{aligned} \phi(\vec{r}, t+dt) &= e^{\mathcal{L} dt} \phi(\vec{r}, t) \\ &= (1 + \mathcal{L} dt) \phi(\vec{r}, t) \\ &= \phi(\vec{r}, t) + dt \mathcal{L} \phi(\vec{r}, t) \end{aligned}$$

(EULER'S METHOD)

AND HIGHER ORDER METHODS ARE ALSO AVAILABLE:

$$\phi(\vec{r}, t+dt) = \left(1 + \mathcal{L} dt + \frac{1}{2} \mathcal{L}^2 dt^2 \right) \phi(\vec{r}, t)$$

IF WE TAKE THE n^{th} ORDER TERM IN dt , WE END UP WITH n^{th} ORDER RUNGE-KUTTA.