

INHOMOGENEOUS PDES (§ 21.5) — GREEN'S FUNCTIONS

SUPPOSE WE HAVE THE PROBLEM

$$\mathcal{L} \phi(\vec{r}) = \rho(\vec{r}) \quad \star$$

↑
COULD HAVE INCLUDED t , OR
ANY LIST OF INDEPENDENT VARIABLES!

→ SUBJECT TO HOMOGENEOUS B.C.'S (FOR NOW).

SUPPOSE WE KNOW $G(\vec{r}, \vec{r}_0)$ SUCH THAT

$$\mathcal{L} G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0),$$

WHERE G SATISFIES OUR HOMOGENEOUS B.C.'S.

WE CAN EXPRESS

$$\begin{aligned} \rho(\vec{r}) &= \int_D \rho(\vec{r}_0) \delta(\vec{r} - \vec{r}_0) dV_0 \\ &= \int_D \rho(\vec{r}_0) [\mathcal{L} G(\vec{r}, \vec{r}_0)] dV_0 \end{aligned}$$

$$\rho(\vec{r}) = \mathcal{L} \left[\int_D \rho(\vec{r}_0) G(\vec{r}, \vec{r}_0) dV_0 \right]$$

SATISFIES \star (AND BC'S!) $\therefore \equiv \phi(\vec{r})$

$$\therefore \phi(\vec{r}) = \int_D \rho(\vec{r}_0) G(\vec{r}, \vec{r}_0) dV_0$$

THE MAIN QUESTION IS HOW TO FIND G !?

- ANSATZ (GUESS)
- FOURIER TRANSFORM (HW 7, #2)
- METHOD OF IMAGES
- EIGENFUNCTION APPROACH (TEDIOUS IN PRACTICE)

EXAMPLE

SUPPOSE $\nabla^2 u = \rho(\vec{r})$, DOMAIN $x > 0$.

AND $u = 0$ ON $x = 0$.

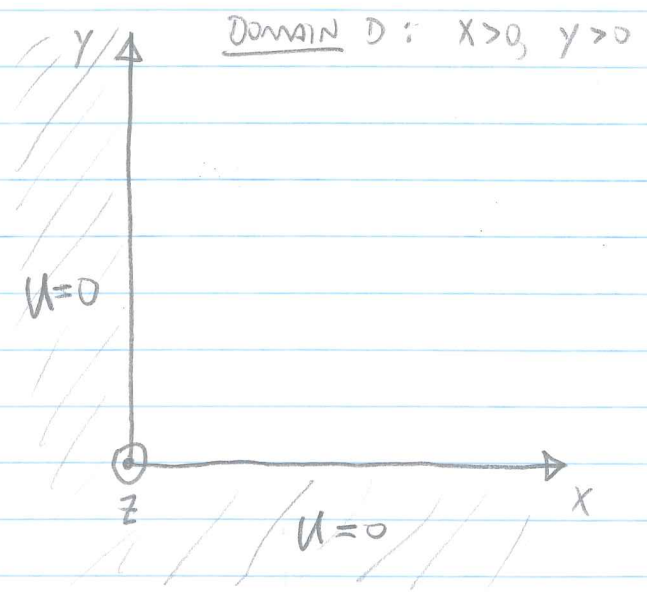
FIND $G(\vec{r}, \vec{r}_0)$ (METHOD OF IMAGES!)

Q: WHAT IF THE B.C.'S WERE: $u = 0$ ON $xy = 0$,

A: FOR A CHARGE AT x_0, y_0 ,
PLACE IMAGE CHARGES AT:

- x_0, y_0
- $x_0, -y_0$
- $-x_0, -y_0$

(WRITE THE G.F. !)



EIGENFUNCTION APPROACH: (§21.5.1)

WE WISH TO SOLVE

$$\mathcal{D} G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0). \quad \star\star$$

SUPPOSE \mathcal{D} , WITH OUR B.C.'S, IS HERMITIAN.

$$\mathcal{D} \psi_n(\vec{r}) = \lambda_n \psi_n(\vec{r}), \quad \langle \psi_n | \psi_m \rangle = \delta_{mn}$$

SINCE THE ψ_n FORM AN ORTHONORMAL BASIS, WE CAN WRITE

$$\delta(\vec{r} - \vec{r}_0) = \sum_m c_m \psi_m(\vec{r})$$

A FUNCTION OF \vec{r}_0 .

WE CAN FIND THE c_m BY THE USUAL TRICK: (TIMES $\psi_n^*(\vec{r})$; $\int dV$)

$$\int_0 dV \psi_n^*(\vec{r}) \delta(\vec{r} - \vec{r}_0) = c_n \int_0 dV \psi_n^* \psi_n$$

$$\Rightarrow c_n = \psi_n^*(\vec{r}_0).$$

SO,

$$\delta(\vec{r} - \vec{r}_0) = \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r}).$$

$$\text{LET } G(\vec{r}, \vec{r}_0) = \sum_n a_n \psi_n(\vec{r})$$

THEN $\star\star$ BECOMES:

$$\mathcal{D} \sum_n a_n \psi_n(\vec{r}) = \sum_n \psi_n^*(\vec{r}_0) \psi_n(\vec{r})$$

$$\Rightarrow a_n \lambda_n = \psi_n^*(\vec{r}_0)$$

$$\Rightarrow \boxed{G(\vec{r}, \vec{r}_0) = \sum_n \frac{1}{\lambda_n} \psi_n^*(\vec{r}_0) \psi_n(\vec{r})} \quad (21.79)$$

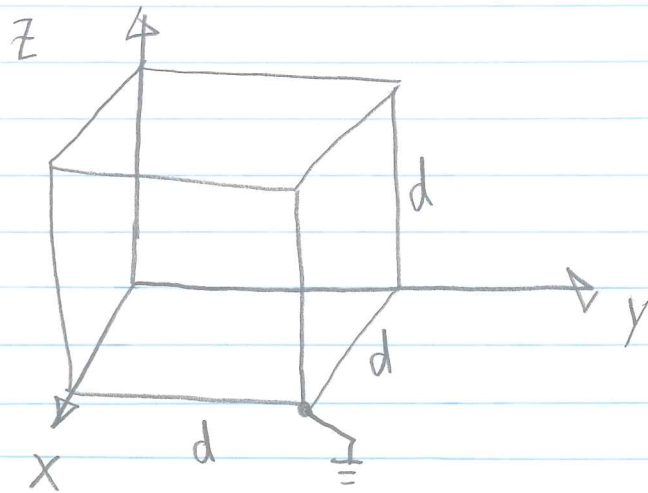
THINGS TO NOTE:

- $G(\vec{r}, \vec{r}_0) = G^*(\vec{r}_0, \vec{r})$
- SAME APPROACH WORKS FOR ODE GREEN'S FUNCTIONS (SEE CH. 17).
- IF \mathcal{D} IS AN OPERATOR WITH CONSTANT COEFFICIENTS, THEN THE ψ_m WILL BE COMPLEX EXPONENTIALS, AND (21.79) IS JUST THE FOURIER TRANSFORM SOLUTION OF THE PROBLEM. (SEE HW 7)

EXAMPLE

$$\nabla^2 \Phi = -\rho(\vec{r})/\epsilon_0$$

INSIDE A CONDUCTIVE, GROUNDED ($\Phi=0$) CUBE:



THE B.C. IS HOMOGENEOUS DIRICHLET.

BUILD G.F. FROM EIGENFUNCTIONS:

$$\nabla^2 \phi_n(\vec{r}) = \lambda_n \phi_n(\vec{r}),$$

$$G(\vec{r}, \vec{r}_0) = \sum_n \frac{\phi_n^*(\vec{r}_0) \phi_n(\vec{r})}{\lambda_n} \quad (21.79)$$

Q: WHAT MUST BE TRUE OF ∇^2 FOR THIS TO WORK?

A: HERMITIAN OVER GIVEN DOMAIN & B.C.'s

[HAVE A STUDENT DO THIS?]

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Now, THE EIGENVALUE PROBLEM:

$$\nabla^2 \phi = \lambda \phi, \quad \phi = 0 \text{ @ } xyz(x-d)(y-d)(z-d) = 0$$

$$\text{SOLN'S: } \phi = A \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

$$\lambda = -(k_x^2 + k_y^2 + k_z^2)$$

B.C.'s:

$$\phi = 0 \text{ @ } x=d \Rightarrow k_x d = \pi l, \quad l=1,2,3$$

AND SIMILARLY IN y, z .

SO

$$\phi_{lmn} = A \sin(l\pi x/d) \sin(m\pi y/d) \sin(n\pi z/d)$$

NORMALIZATION (NEVER FORGET THIS STEP!)

$$\int_0^d \int_0^d \int_0^d \phi^* \phi \, dx \, dy \, dz = 1$$

$$A^2 \left(\frac{d}{2}\right)^3 = 1 \Rightarrow A = \left(\frac{2}{d}\right)^{3/2}$$

$$\phi_{lmn} = \left(\frac{2}{d}\right)^{3/2} \sin(l\pi x/d) \sin(m\pi y/d) \sin(n\pi z/d)$$

$$l, m, n \in \{1, 2, 3, \dots\}$$

$$\lambda_{lmn} = -k^2 = -\frac{\pi^2}{d^2} (l^2 + m^2 + n^2).$$



THEN,

$$G(\vec{r}, \vec{r}_0) = \sum_{l,m,n} \frac{-\phi_{l,m,n}(\vec{r}) \phi_{l,m,n}(\vec{r}_0)}{\frac{\pi^2}{d^2}(l^2+m^2+n^2)}$$

$$= \sum_{l,m,n} \frac{-8 \sin(l\pi x_0/d) \sin(l\pi y_0/d) \sin(l\pi z_0/d)}{\pi^2 d (l^2+m^2+n^2)} \sin(l\pi x/d) \sin(l\pi y/d) \sin(l\pi z/d).$$

Q: HOW MANY TERMS DOES IT TAKE TO CONVERGE TO ~ 1%?

A: $\approx 10^3$ TERMS: $l = 1..10, m = 1..10, n = 1..10$

Q: FOR 1% ACCURACY, ^{ESTIMATING $\Phi(\vec{r})$} ON A 100 x 100 x 100 GRID, ^{1% SPATIAL RESOLUTION} HOW MANY TIMES WOULD YOU HAVE TO CALL THE SIN FUNCTION?

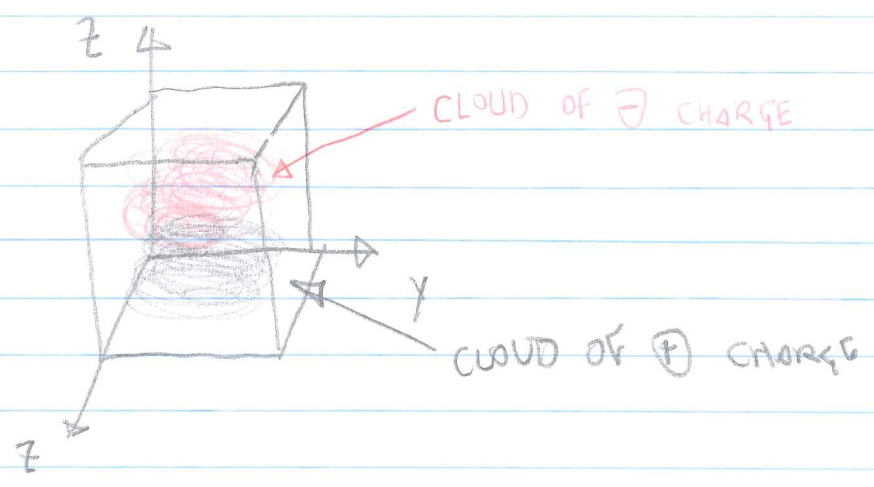
A: 6 calls x 10^3 TERMS x 10^6 GRID POINTS.
= 6×10^9 . OUCH!

STILL, THE COST OF A GIGAFLOP IS SMALL, SO THIS IS NOT SO BAD FOR NUMERICAL USE.

Q: GIVEN THE SLOW CONVERGENCE OF THE EIGENFUNCTION EXPANSION, IS IT GOOD FOR ANYTHING?

A: SURE. IF $\rho(\vec{r})$ IS SIMPLE ENOUGH IN TERMS OF ϕ_{lmn} , THEN IT'S GREAT. FOR EXAMPLE...

SUPPOSE: $\rho(\vec{r}) = \rho_0 \sin(\pi x/d) \sin(\pi y/d) \sin(2\pi z/d)$



Q: NOW, WHAT IS $\Phi(\vec{r})$?

A: BEGIN BY WRITING

$$-\frac{\rho(\vec{r})}{\epsilon_0} = -\frac{\rho_0}{\epsilon_0} \left(\frac{d}{2}\right)^{3/2} \phi_{112}(\vec{r})$$

SO,

$$\Phi(\vec{r}) = + \int_V \sum_{lmn} \frac{\phi_{lmn}(\vec{r}) \phi_{lmn}^*(\vec{r}_0) \phi_{112}(\vec{r}_0)}{\left(\frac{\pi^2}{d^2}\right) (l^2 + m^2 + n^2)} \left(\frac{d}{2}\right)^{3/2} \frac{\rho_0}{\epsilon_0} dV_0$$

SINCE $\langle \phi_{lmn} | \phi_{l'm'n'}^* \rangle = \delta_{ll'} \delta_{mm'} \delta_{nn'}$, THE SUM GOES AWAY AND $l, m, n = 1, 1, 2 \dots$



$$\Phi(\vec{r}) = \frac{d^2}{6\pi^2} \frac{\rho_0}{\epsilon_0} \sin(\pi x/d) \sin(\pi y/d) \sin(2\pi z/d)$$

NOW WE SEE THE POWER OF THE EIGENVALUE APPROACH.

NEXT TIME:

INHOMOGENEOUS B.C.'S