

SO FAR IN CH. 21, PDES

- SEPARATION OF VARIABLES
- INTEGRAL TRANSFORMS
- GREEN'S FUNCTIONS FOR HOMOGENEOUS B.C.'S
 - METHOD OF IMAGES
 - METHOD OF EIGENFUNCTION EXPANSION

TODAY — § 21.5.2, GREEN'S FUNCTIONS FOR
INHOMOGENEOUS B.C.'S

IT'S A THEORY DAY — NO EXAMPLES!

UPCOMING — § 21.5.3 G.F. FOR DIRICHLET B.C.'S

§ 21.5.4 G.F. FOR NEUMANN B.C.'S

- AND THAT'S THE END OF THE CHAPTER!
- KNOWING HOW TO SOLVE INHOMOGENEOUS PDES WITH INHOMOGENEOUS B.C.'S IS THE PINNACLE OF THE THEORY FOR LINEAR PDES.
- YOU MAY FIND IN THE RECENT LITERATURE G.F. SOLUTIONS OF THIS TYPE FOR PROBLEMS SUCH AS

$$\nabla \times \vec{B} = \alpha \vec{B},$$

WITH $\vec{B} \cdot \hat{n}$ GIVEN ON THE BOUNDARY.

PDE GREEN'S FUNCTIONS FOR INHOMOGENEOUS B.C.'S

(§ 21.5.2)

WE'LL FOCUS ON POISSON'S EQUATION, AS DOES RHB.

TO BE SOLVED:

- $\nabla^2 u(\vec{r}) = \rho(\vec{r})$, $\vec{r} \in V$ BOUNDED BY S' *
- INHOMOGENEOUS NEUMANN OR DIRICHLET B.C.'S ON S'
- THE GENERAL APPROACH WORKS FOR OTHER OPERATORS, BUT WE FOCUS ON ∇^2 .
- MY PRESENTATION DIFFERS SOMEWHAT FROM RHB.

BEGIN BY SOLVING

$$\nabla^2 G(\vec{r}, \vec{r}_0) = \delta(\vec{r} - \vec{r}_0); \quad \vec{r}, \vec{r}_0 \in V \quad **$$

- DO NOT SUBJECT G TO THE B.C.'S
- G IS THEREFORE NOT UNIQUE.

START WITH THE SUPERPOSITION

$$\int_V G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) dV_0 \quad \neq \quad \underline{\underline{u(\vec{r})}}$$

- DOESN'T SATISFY B.C.'S, EVEN IF G DOES, BECAUSE B.C.'S ARE INHOMOGENEOUS.
- DOES SATISFY THE PDE $\nabla^2 u = \rho$. (*)

NOW SEEK A CORRECTION TERM $h(\vec{r})$ SUCH THAT

$$u(\vec{r}) = \int_V G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) dV_0 + h(\vec{r}) \quad (*)$$

Q: SPECULATE: HOW CAN WE CHOOSE G AND h ?

A: ① LET G SATISFY HOMOGENEOUS B.C.'S, SO $\int G \rho dV_0$ DOES TOO.

② LET $h(\vec{r})$ SATISFY $\nabla^2 h = 0$

③ LET $h(\vec{r})$ SATISFY THE INHOMOGENEOUS B.C.'S FOR $u(\vec{r})$.

WE ASSUME NONE OF THIS, BUT IT WILL FALL OUT...

FIRST, EXPRESS $u(\vec{r})$ (TRIVIALY!) AS

$$u(\vec{r}) = \int_V u(\vec{r}_0) \delta(\vec{r} - \vec{r}_0) dV_0$$

SO THAT IT CAN BE PULLED INTO THE INTEGRAL WITH G . (*) BECOMES:

$$\begin{aligned}
 h(\vec{r}) &= \int_V \left[-G(\vec{r}, \vec{r}_0) \underbrace{\rho(\vec{r}_0)}_{\downarrow \star} + u(\vec{r}_0) \underbrace{\delta(\vec{r} - \vec{r}_0)}_{\downarrow \star\star} \right] dV_0 \\
 &= \int_V \left[-G(\vec{r}, \vec{r}_0) \nabla_0^2 u(\vec{r}_0) + u(\vec{r}_0) \nabla^2 G(\vec{r}, \vec{r}_0) \right] dV_0
 \end{aligned}$$

STRATEGY

IF WE CAN CHANGE THIS ∇^2 TO ∇_0^2 , THEN

WE'LL HAVE THE LHS OF GREEN'S 2nd THEOREM.

THIS WILL GIVE US A SURFACE INTEGRAL (ON S'),

WHICH WILL CONNECT US TO THE B.C.'S.



TAKE ~~AA~~ AND SWITCH VARIABLES, $\vec{r} \leftrightarrow \vec{r}_0$:

$$\nabla_0^2 G(\vec{r}_0, \vec{r}) = \delta(\vec{r}_0 - \vec{r})$$

BUT δ IS SYMMETRIC ABOUT THE ORIGIN, SO

$$\nabla_0^2 G(\vec{r}_0, \vec{r}) = \delta(\vec{r} - \vec{r}_0) = \nabla^2 G(\vec{r}, \vec{r}_0)$$

SUPPOSE G SATISFIES HOMOGENEOUS B.C.'S, SO THAT ∇^2 IS HERMITIAN IN ~~AA~~ (THOUGH NOT IN ~~AA~~!)

$$\Rightarrow G(\vec{r}_0, \vec{r}) = G^*(\vec{r}, \vec{r}_0) \quad (\text{RESULT OF LAST LECTURE})$$

AND THUS

$$\nabla_0^2 G^*(\vec{r}, \vec{r}_0) = \nabla^2 G(\vec{r}, \vec{r}_0)$$

ASSUME FURTHER $G \in \mathbb{R}$ (TRUE IF THE OPERATOR, $\rho(\vec{r})$, AND B.C.'S $\in \mathbb{R}$). THEN WE HAVE

$$\nabla_0^2 G(\vec{r}, \vec{r}_0) = \nabla^2 G(\vec{r}, \vec{r}_0) \quad !!!$$

SO THEN

$$h(\vec{r}) = \int_V [u(\vec{r}_0) \nabla_0^2 G(\vec{r}, \vec{r}_0) - G(\vec{r}, \vec{r}_0) \nabla_0^2 u(\vec{r}_0)] dV_0$$

N.B.: OUR PRESENTATION SO FAR IS GENERIC. UP TO THIS POINT, ∇_0^2 COULD BE ANY REAL OPERATOR \mathcal{D} , WITH G ANY GREEN'S FUNCTION FOR \mathcal{D} SATISFYING HOMOGENEOUS B.C.'S, SO LONG AS \mathcal{D} BECOMES HERMITIAN WHEN HOMOGENEOUS B.C.'S ARE APPLIED.

NOW, WE GET SPECIFIC TO $\nabla \cdot$. USING GREEN'S SECOND THEOREM,

$$h(\vec{r}) = \oint_{S'} \left[u(\vec{r}_0) \nabla_0 G(\vec{r}, \vec{r}_0) - G(\vec{r}, \vec{r}_0) \nabla_0 u(\vec{r}_0) \right] \cdot \hat{n}_0 dA_0$$

$$= \oint_{S'} \left[u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} - G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0} \right] dA_0$$

THEREFORE, GOING BACK TO \otimes ,

$$u(\vec{r}) = \int_V G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) dV_0 + \oint_{S'} \left[u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} - G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0} \right] dA_0$$

CF. EQ. (21.84).

BEFORE WE GO FURTHER, RECALL THAT WE HAD MADE 3 PREDICTIONS ABOUT G & h TO SOLVE THIS PROBLEM:

- ① LET G SATISFY HOMOGENEOUS B.C.'S \leftarrow THIS WAS USED IN OUR DERIVATION OF $h(\vec{r})$, BUT THERE ARE OTHER APPROACHES.
- ② LET $h(\vec{r})$ SATISFY $\nabla^2 h = 0$ ON V (INSIDE S).
- ③ LET $h(\vec{r})$ SATISFY THE INHOMOGENEOUS B.C.'S.

Q: CAN WE AGREE THAT THESE CONDITIONS LOGICALLY ENTAIL \otimes ? DISCUSS.

A: YES, THEY DO.



CASE I: DIRICHLET B.C.'s, $u(\vec{r}) = \phi(\vec{r})$ FOR $\vec{r} \in S$

- ① LET $G(\vec{r}, \vec{r}_0) = 0$ FOR $\vec{r}_0 \in S$ (WHY? LOOK AT 21.84).
IT ALSO FOLLOWS THAT $G = 0$ FOR $\vec{r} \in S$ (WHY?)

$$\text{THEN } h(\vec{r}) = \oint_S [u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0}] dA_0$$

$$\begin{aligned} \text{② } \nabla^2 h &= \left. \begin{aligned} &\oint_S [u(\vec{r}_0) \frac{\partial}{\partial n_0} \nabla^2 G(\vec{r}, \vec{r}_0)] dA_0 \\ &= \oint_S [u(\vec{r}_0) \frac{\partial}{\partial n_0} \delta(\vec{r} - \vec{r}_0)] dA_0 \end{aligned} \right\} \begin{array}{l} \text{MIGHT BE PEDAGOGICALLY} \\ \text{USEFUL TO TAKE IT ANOTHER} \\ \text{WAY: } \frac{\partial}{\partial n_0} \delta(\vec{r} - \vec{r}_0) \end{array} \end{aligned}$$

• NOTE THAT $\delta(\vec{r} - \vec{r}_0) = 0$ FOR $\vec{r} \neq \vec{r}_0$.

• THUS, $\nabla^2 h(\vec{r}) = 0$ FOR \vec{r} INSIDE (NOT ON) S . ✓

- ③ DOES $h(\vec{r}) = \phi(\vec{r})$ FOR $\vec{r} \in S$?

$$\begin{aligned} h(\vec{r}) &= \oint_S [\phi(\vec{r}_0) \nabla_0 G(\vec{r}, \vec{r}_0)] \cdot \hat{n} dA_0 \\ &= \int_V \nabla_0 \cdot [\phi(\vec{r}_0) \nabla_0 G(\vec{r}, \vec{r}_0)] dV_0 \quad (\text{GAUSS'S THM}) \\ &= \int_V \left\{ [\nabla_0 \phi(\vec{r}_0)] \cdot [\nabla_0 G(\vec{r}, \vec{r}_0)] \right. \\ &\quad \left. + \phi(\vec{r}_0) \underbrace{\nabla_0^2 G(\vec{r}, \vec{r}_0)}_{\delta(\vec{r} - \vec{r}_0)} \right\} dV_0 \\ &= \phi(\vec{r}) + \int_V \underbrace{[\nabla_0 \phi(\vec{r}_0)] \cdot [\nabla_0 G(\vec{r}, \vec{r}_0)]}_{\uparrow} \end{aligned}$$

Q: WHAT ABOUT THE SECOND TERM? ↑

A: SINCE $\phi(\vec{r}_0)$ IS DEFINED ONLY FOR $\vec{r}_0 \in S$, WE CAN ONLY EVALUATE THE TANGENTIAL ($\perp \hat{n}_0$) COMPONENTS, $\nabla_0 \phi(\vec{r}_0) \rightarrow \nabla_0^{\parallel} \phi(\vec{r}_0)$. NOW,

$$\nabla_0^{\parallel} G(\vec{r}, \vec{r}_0) = 0 \text{ BECAUSE OF THE B.C.}$$

(G DOESN'T VARY OVER THE SURFACE!)

$$\stackrel{\text{SO}}{\Rightarrow} [\nabla_0 \phi(\vec{r}_0)] \cdot [\nabla_0 G(\vec{r}, \vec{r}_0)] = 0 \text{ ON } S.$$

$$\therefore h(\vec{r}) = \phi(\vec{r}) \text{ ON } S. \quad \checkmark$$

FOR AN ADDED EXERCISE, YOU MAY SHOW THAT PROPOSITIONS ①, ② & ③ ARE SATISFIED WHEN THE B.C.'S ARE NEUMANN.

GENERAL RULE: (§§ 21.5.3 - 21.5.4)

- FOR DIRICHLET CONDITIONS $u(\vec{r}) = \phi(\vec{r})$, $\vec{r} \in S$,
USE $G(\vec{r}, \vec{r}_0) = 0$ FOR $\vec{r}_0 \in S$
- FOR NEUMANN CONDITIONS $\frac{\partial}{\partial n} u(\vec{r}) = \psi(\vec{r})$, $\vec{r} \in S$,
USE $\frac{\partial}{\partial n} G(\vec{r}, \vec{r}_0) = 0$ FOR $\vec{r}_0 \in S$

~~Q: COULD WE HAVE ANTICIPATED THIS BASED ON~~