

INHOMOGENEOUS B.C.'s § § 21.5.2-21.5.4

GIVEN  $\nabla^2 u = \rho(\vec{r})$ , WITH DIRICHLET OR NEUMANN B.C.'s ON S ENCLOSING DOMAIN V, OUR SOLUTION IS:

$$u(\vec{r}) = \int_V G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) dV_0 + \int_S \left[ u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} - G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0} \right] dA_0, \\ \nabla^2 G = \delta(\vec{r} - \vec{r}_0)$$

DIRICHLET

NEUMANN

$u = \phi(\vec{r}), \vec{r} \in S$

$\frac{\partial u}{\partial n} = \psi(\vec{r}), \vec{r} \in S$

WANT  $G(\vec{r}, \vec{r}_0) = 0, \vec{r}_0 \in S$

WANT  $\frac{\partial}{\partial n_0} G(\vec{r}, \vec{r}_0) = 0, \vec{r}_0 \in S$

GIVEN  $F(\vec{r}, \vec{r}_0)$  (FUND. SOLN.) SUCH THAT  $\nabla^2 F = \delta(\vec{r} - \vec{r}_0)$ , LET  $G = F + H$  WHERE  $\nabla^2 H = 0$ , AND

$F + H = 0, \vec{r}_0 \in S$

$\frac{\partial}{\partial n} (F + H) = 0, \vec{r}_0 \in S$

THEN

THEN

$u = \int_V G \rho(\vec{r}_0) dV_0 + \int_S \phi(\vec{r}_0) \frac{\partial G}{\partial n_0} dA_0$

$u = \int_V G \rho(\vec{r}_0) dV_0 - \int_S G(\vec{r}, \vec{r}_0) \psi(\vec{r}_0) dA_0$

(DISCUSS THE LOGIC OF THE METHOD OF IMAGES, F+H)

NOTE THAT (21.84) IS USEFUL FOR SOLUTION OF THE HOMOGENEOUS EQUATION,

$$\nabla^2 u = 0, \quad \vec{r} \in V,$$

WITH INHOMOGENEOUS B.C.'S ON  $S'$  ENCLOSING  $V$ .

$$\Rightarrow u(\vec{r}) = \oint_{S'} [u(\vec{r}_0) \frac{\partial G}{\partial n_0} - G \frac{\partial u(\vec{r}_0)}{\partial n_0}] dA_0$$

HOW PECULIAR! IT APPEARS THAT INHOMOGENEITY ON THE BOUNDARY CAN BE HANDLED JUST LIKE A SOURCE TERM IN THE INTERIOR, USING A G.F.!

EXAMPLE

GIVEN ELECTRICAL POTENTIAL  $\Phi(x, y, 0) = V(x, y)$  AND  $\Phi(\vec{r}) \rightarrow 0$  AS  $z \rightarrow \infty$ , FIND  $\Phi(\vec{r})$ .

$$\nabla^2 \Phi = 0, \quad F(\vec{r}, \vec{r}_0) = \frac{1}{4\pi|\vec{r} - \vec{r}_0|}$$

BY 21.84,

$$\Phi(\vec{r}) = \oint_{S'} [u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} - G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0}] dA_0$$

NOW, SINCE B.C.'S ARE DIRICHLET, CHOOSE

$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} - \frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2}}$$

$\swarrow H(\vec{r}, \vec{r}_0)$

SO THAT  $G=0$  ON  $S'$ , AND 2<sup>nd</sup> TERM VANISHES!

$$\text{NOW, } \Phi(\vec{r}) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 \left( V(x_0, y_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial z_0} \Big|_{z_0=0} \right)$$

(DISCUSS THE SET-UP!)

Q: IS THIS A CLOSED SURFACE, A LA  $\oint_{S'}$ ? ...

SO, WE NEED

$$\left. \frac{\partial G}{\partial z_0} \right|_{z_0=0} = \left[ \frac{z-z_0}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} - \frac{-(z+z_0)}{4\pi [(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2]^{3/2}} \right]_{z_0=0}$$

$$= \frac{z}{2\pi [(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}}$$

$$\Phi(\vec{r}) = \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dy_0 \left[ \frac{V(x_0, y_0) z}{2\pi [(x-x_0)^2 + (y-y_0)^2 + z^2]^{3/2}} \right]$$

$$= V(x, y) * \frac{z}{2\pi [x^2 + y^2 + z^2]^{3/2}},$$

WHERE \* IS CONVOLUTION IN X, Y ONLY.

2D THE FOURIER TRANSFORM IS INTERESTING:  $\tilde{F}_{xy}$

$$\tilde{\Phi}(k_x, k_y, z) = \tilde{V}(k_x, k_y) \tilde{F}_{xy} \left[ \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

NEXT  $\rightarrow$

$\rightarrow$

THE 2D FOURIER TRANSFORM OF  $\Phi$  IS INTERESTING.  
CONVOLUTION THEOREM:

$$\mathcal{F}_x (f(x) * g(x)) = \sqrt{2\pi} \tilde{f}(k) \tilde{g}(k)$$

$$\Rightarrow \mathcal{F}_{xy} (f(x,y) * g(x,y)) = 2\pi \tilde{f}(k_x, k_y) \tilde{g}(k_x, k_y)$$

So

$$\tilde{\Phi}(k_x, k_y, z) = \tilde{V}(k_x, k_y) \mathcal{F}_{xy} \left[ \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right]$$

NOW, EQ. 13.52 HAS:

$$\mathcal{F}_{xy} [f(\rho)] = \int_0^{\infty} f(\rho) J_0(k\rho) \rho d\rho, \quad k \equiv \sqrt{k_x^2 + k_y^2}$$

I ALSO HAVE, FROM BRACEWELL, THE HANKEL TRANSFORM:

$$F(q) \equiv 2\pi \int_0^{\infty} f(\rho) J_0(2\pi q\rho) \rho d\rho$$

AND

$$F(q) = \frac{2\pi}{a} e^{-2\pi a q} \quad \text{FOR } f(\rho) = \frac{1}{(a^2 + \rho^2)^{3/2}}.$$



IN GENERAL, WHAT IS  $\tilde{\mathcal{F}}_{xy}[f(\rho)]$  IN TERMS OF  $F(q)$ ?

WELL, LET  $2\pi q = k$

$$F(q) = 2\pi \int_0^\infty f(\rho) J_0(k\rho) \rho d\rho$$

$$= 2\pi \tilde{f}(k)$$

$$\therefore \tilde{f}(k) = \frac{1}{2\pi} F\left(\frac{k}{2\pi}\right)$$

SO

$$\tilde{\mathcal{F}}\left[\frac{z}{(\rho^2+z^2)^{3/2}}\right] = \frac{z}{z} e^{-kz}$$

WE THUS HAVE:

$$\tilde{\Phi}(k_x, k_y, z) = \tilde{V}(k_x, k_y) e^{-kz}$$

Q: WHY IS THIS AN ESPECIALLY PRACTICAL RESULT?

WE'VE ACTUALLY DONE THIS BEFORE, BY FOURIER TRANSFORMING THE PDE  $\nabla^2 \Phi = 0$ , AND USING SUPERPOSITION TO BUILD THE RESULT AT THE BOUNDARY.