

### COMMENT ON § 21.5.4

IN THE NEUMANN PROBLEM,

$$\nabla^2 u = \rho(\vec{r}) \quad \text{ON } V \text{ BOUNDED BY } S,$$

$$\frac{\partial}{\partial n} u(\vec{r}) = \psi(\vec{r}) \quad \text{ON } S.$$

WE HAVE HERE A POSSIBLE INCONSISTENCY:

$$\int_V \rho(\vec{r}) dV = \int_V \nabla \cdot (\nabla u) dV$$

$$= \oint_S \hat{n} \cdot (\nabla u) dA \quad (\text{GAUSS'S THM})$$

$$\int_V \rho(\vec{r}) dV = \oint_S \psi(\vec{r}) dA$$

REQUIREMENT FOR CONSISTENCY.

THUS,  $\psi(\vec{r})$  CANNOT BE QUITE ARBITRARY!

THIS BECOMES A SERIOUS CONCERN WHEN WE TRY TO CONSTRUCT A GREEN'S FXN!



(2)

OUR USUAL APPROACH, STARTING FROM EQ (21.84), WOULD BE:

$$u(\vec{r}) = \int_V G(\vec{r}, \vec{r}_0) \rho(\vec{r}_0) dV_0 \\ + \oint_S \left[ u(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} - G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0} \right] dA_0,$$

CHOOSE  $G(\vec{r}, \vec{r}_0)$  SATISFYING

$$\nabla^2 G = \delta(\vec{r}, \vec{r}_0)$$

AND LET  $\frac{\partial G}{\partial n_0} = 0$  ON  $S$ .

WHAT'S THE PROBLEM HERE?

$$\rightarrow \text{IF } \nabla^2 G = \nabla \cdot (\nabla G) = \delta(\vec{r}, \vec{r}_0)$$

$$\text{THEN } \int_V \nabla^2 G dV = 1$$

$$1 = \oint_S \hat{n} \cdot \nabla G dA = \oint_S \frac{\partial G}{\partial n} dA$$

$$1 = 0 \quad \underline{\text{OOPS!}}$$

→

RHB PROPOSE THE FOLLOWING SOLUTION:

$$\left. \frac{\partial G}{\partial n} \right|_{\vec{r} \in S} = \frac{1}{A}, \quad A \equiv \oint_S dA$$

Q: IS THE CONSISTENCY REQUIREMENT SATISFIED?

$$\begin{aligned} \text{A: } \int_V \nabla^2 G dV &= 1 = \oint_S \frac{\partial G}{\partial n} dA \\ &= \frac{1}{A} \oint_S dA = 1 \quad \checkmark \end{aligned}$$

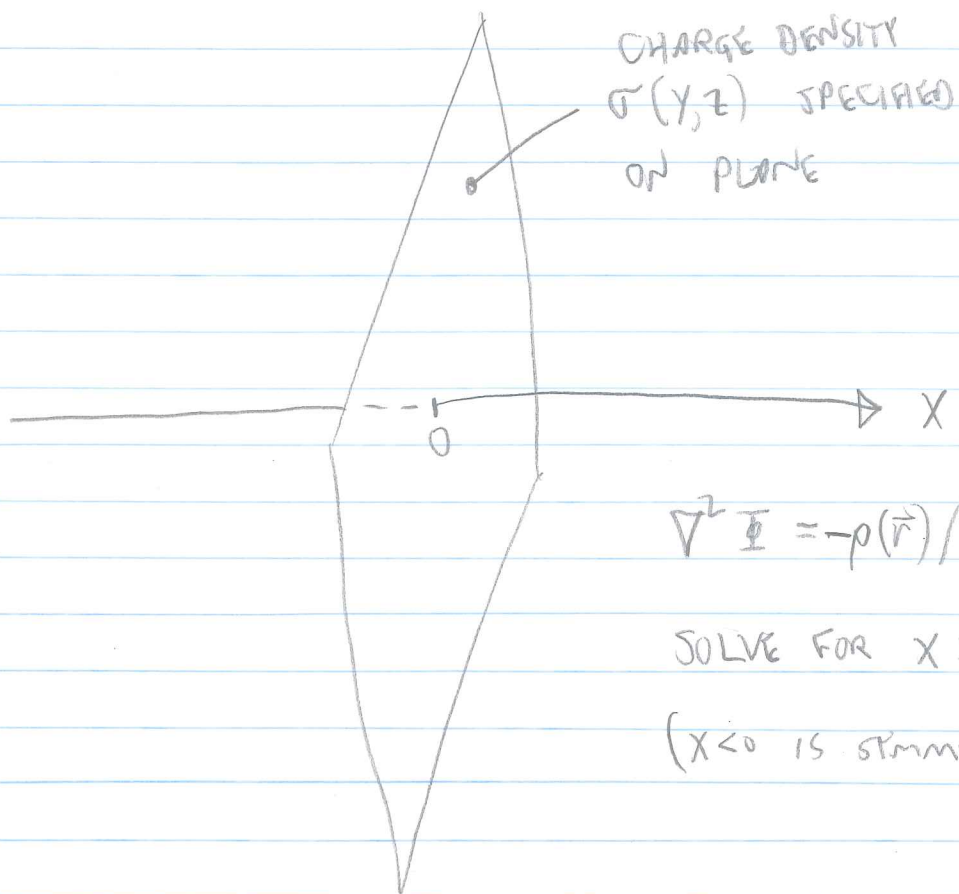
COMMENT: IT CAN BE VERY DIFFICULT TO FIND SUCH A GREEN'S FUNCTION. HOWEVER, A NICE EXAMPLE IS GIVEN.

NOW, EQ. (21.84) BECOMES:

$$\begin{aligned} U(\vec{r}) &= \int_V G \rho(\vec{r}_0) dV_0 + \oint_S \left[ \frac{U(\vec{r}_0)}{A} - G \psi(\vec{r}_0) \right] dA_0 \\ &= \int_V G \rho(\vec{r}_0) dV_0 + \underbrace{\langle U \rangle_S}_{\uparrow} - \oint_S G \psi(\vec{r}_0) dA_0. \end{aligned}$$

NOTE THAT THIS IS AN ARBITRARY CONSTANT.

I WILL NOT EMPHASIZE THIS APPROACH. THE TRICK ISN'T GENERICALLY USEFUL, AND B.C.'S OF THIS FORM ARE NOT SO COMMON (IMPOSING  $\frac{\partial U}{\partial n}$  IS UNPHYSICAL!)



PHYSICS TELLS US:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} dV_0$$

$$= \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dy_0 \int_{-\infty}^{\infty} dz_0 \frac{\sigma(y_0, z_0)}{\sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}}$$

WE CAN USE THIS AS A TEST OF OUR ABILITY TO HANDLE NEUMANN B.C.'S :



$$\oint_{\partial V} \vec{E} \cdot d\vec{A} = Q_{enc}/\epsilon_0$$

$$2E_n(y,z)A = (\sigma(y,z)/\epsilon_0)A$$

$$E_n(y,z) = \frac{\sigma(y,z)}{2\epsilon_0} = \frac{+\partial\Phi}{\partial n}$$



GREEN'S FUNCTION: USE POSITIVE MIRROR CHARGE (WHY?)

$$\nabla^2 G = \delta(\vec{r} - \vec{r}_0)$$

$$G = F + H \quad \text{WHY BOTH TERMS} < 0 \text{ ? (DISCUSS)}$$

$$= \frac{-1}{4\pi|\vec{r} - \vec{r}_0|} + \frac{-1}{4\pi\sqrt{(x+x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

Q: IS THERE ANY SELF-CONSISTENCY ISSUE?

A: NO. IMPLIED DIRICHLET B.C.  $\Phi \rightarrow 0$  AS  $|x| \rightarrow \infty$ .

NOW, EQ. 21.84 BECOMES:

$$u(\vec{r}) = \int_V G \rho dV_0 - \int_S G(\vec{r}, \vec{r}_0) \frac{\partial u(\vec{r}_0)}{\partial n_0} dA_0.$$

$$= - \int_S \frac{\sigma(y_0, z_0)}{2\epsilon_0} \left[ \frac{-2}{4\pi\sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}} \right] dA_0.$$

$$= \frac{+1}{4\pi\epsilon_0} \int_{-a}^a dz_0 \int_{-a}^a dy_0 \frac{\sigma(y_0, z_0)}{\sqrt{x^2 + (y-y_0)^2 + (z-z_0)^2}}$$



LOOKS LIKE PHYSICS!