

SCALAR DIFFRACTION THEORY

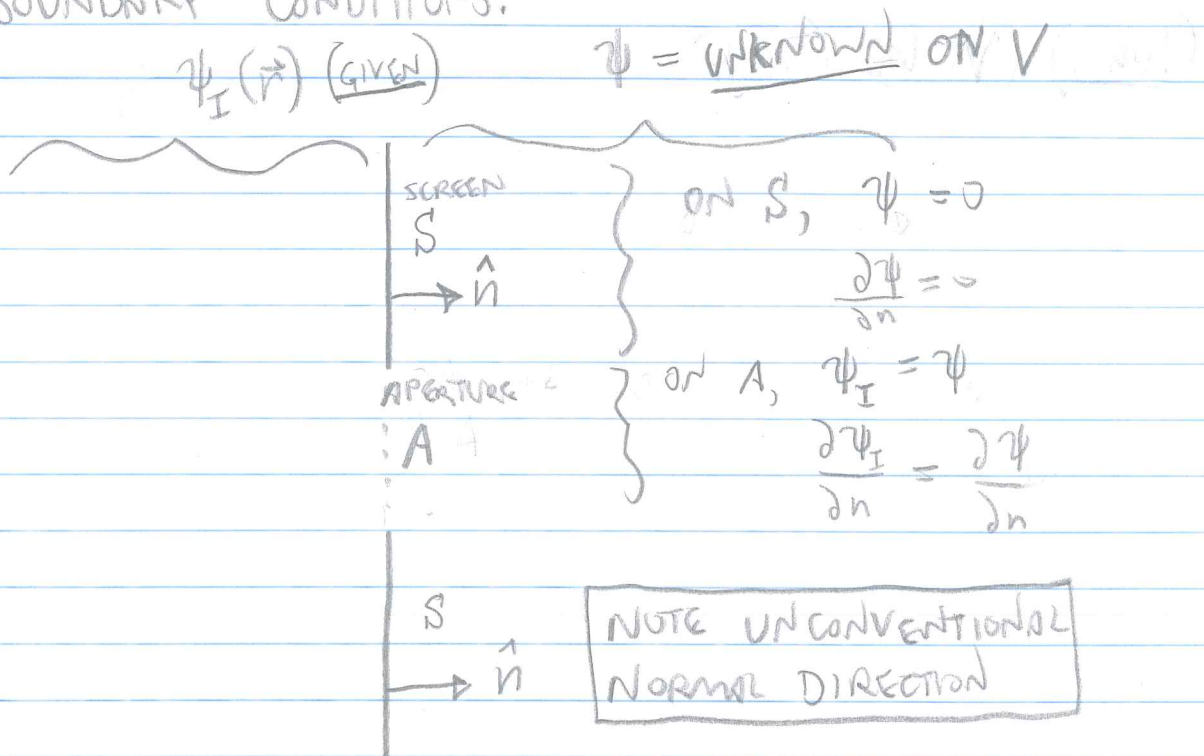
$$c^2 \nabla^2 u = \frac{\partial^2}{\partial t^2} u$$

LET $u = \psi(\vec{r}) e^{i\omega t}$,
 $k = \omega/c$

MIGHT STILL
 BE LURKING
 MINUS SIGN ERRORS

THEN $(\nabla^2 + k^2)\psi = 0$. \star
 HELMHOLTZ EQN.

BOUNDARY CONDITIONS:



THIS B.C. IS CALLED THE KIRCHOFF APPROXIMATION.
 IT IS, IN FACT, MATHEMATICALLY INCONSISTENT AND
UNPHYSICAL! ... DISCUSS!

USE $G(\vec{r}, \vec{r}') = \frac{-e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$ (GREEN'S FUNCTION)

SHOW THAT

$$(\nabla^2 + k^2)G = \delta(\vec{r} - \vec{r}'). \quad \star \star$$

(TIME PERMITTING)

USE GREEN'S 2nd THEOREM:

$$\int_V (G \nabla^2 \psi - \psi \nabla^2 G) dV$$

$$= - \int_{S'} (G \nabla \psi - \psi \nabla G) \cdot d\vec{A}$$

$$- \int_A (G \nabla \psi - \psi \nabla G) \cdot d\vec{A}$$

$$- \int_{S''} (G \nabla \psi - \psi \nabla G) \cdot d\vec{A}$$

MINUS SINCE \hat{n} DIRECTED INWARD (USUAL DIFFRACTION CONVENTION!)

IN LHS, SUBSTITUTE \star AND $\star\star$:

$$\int_V (-G \cancel{\nabla^2 \psi} + \psi \cancel{\nabla^2 G} - \psi \delta(\vec{r} - \vec{r}')) dV$$

$$\rightarrow -\psi(\vec{r}')$$

ON RHS, $\int_{S'} \rightarrow 0$, $\int_{S''} \rightarrow 0$

(THE LATTER IS HARDER THAN YOU MIGHT SUPPOSE!)

WE ARE LEFT WITH

$$\psi(\vec{r}') = \int_A (G(\vec{r}, \vec{r}') \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla G(\vec{r}, \vec{r}')) \cdot d\vec{A}$$

SINCE $G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})$

$$\psi(\vec{r}) = \int_A \left[G(\vec{r}, \vec{r}') \frac{\partial \psi_I(\vec{r}')}{\partial n'} - \psi_I(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \right] dA'$$

KIRCHOFF'S DIFFRACTION FORMULA.

LAST TIME, FOUND KIRCHOFF'S FORMULA,

$$\psi(\vec{r}) = \int_A \left[G(\vec{r}, \vec{r}') \frac{\partial \psi_I(\vec{r}')}{\partial n'} - \psi_I(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \right] dA' \quad \star$$

LET $G(\vec{r}, \vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$

- APERTURE A IN z=0 PLANE,
- $\psi_I(\vec{r}) = e^{ikz}$

Q: WHAT DOES \star BECOME?

$$\psi(\vec{r}) = -\frac{1}{4\pi} \int_A \frac{e^{ikR}}{R} \left[\frac{\partial \psi_I}{\partial n'} + ik \left(1 + \frac{i}{kR} \right) \frac{z}{R} \psi_I \right] dA'$$

\uparrow
 ik
 $\cos \theta$

A:
$$\psi(\vec{r}) = -\frac{1}{4\pi} \int_A \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \left[ik(1 + \cos \theta) - \frac{\cos \theta}{|\vec{r}-\vec{r}'|} \right] dA'$$

LET'S TRY TO REMOVE SOME OF THE INCONSISTENCIES OF KIRCHOFF'S APPROACH,

WITHOUT APPROXIMATIONS:

$$\int_V (G \nabla'^2 \psi - \psi \nabla'^2 G) dV' = - \oint (G \nabla' \psi - \psi \nabla' G) \cdot d\vec{A}'$$

RECALL $\nabla'^2 \psi = -k^2 \psi$

$$\nabla'^2 G = -k^2 G + \delta(\vec{r}, \vec{r}')$$

WE THEREFORE HAVE

$$\psi(\vec{r}) = \oint (G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'}) dA'$$

Q: WHAT IS ψ IN THIS CONTEXT?

A: TOTAL ψ ON BOUNDARY, NOT JUST ψ_I .

OF COURSE,

$$\oint = \int_S + \int_A + \int_{S_\infty}$$

0 AS $r' \rightarrow \infty$

$$\psi(\vec{r}') = \left\{ \int_S + \int_A \right\} (G \frac{\partial \psi}{\partial n'} - \psi \frac{\partial G}{\partial n'}) dA'$$

RAYLEIGH-SOMMERFELD DIFFRACTION THEORY *

WORKS ONLY FOR:
 $A, S @ z=0$

DIRICHLET

LET $\psi=0$ ON SCREEN S' , AND
 $\psi=\psi_I$ ON APERTURE A .

CHOOSE A $G_D(\vec{r}, \vec{r}') = 0$ ON S' , SO $G \frac{\partial \psi}{\partial n}$ TERM VANISHES...

Q: HOW?

A: METHOD OF IMAGES.

$$G_D(\vec{r}, \vec{r}') = \frac{-1}{4\pi} \left(\frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'} \right)$$

WHERE $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$,

$$R' = \sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}$$

$$\left. \begin{aligned} \frac{\partial R}{\partial n'} &= \frac{\partial R}{\partial z'} = \frac{z'-z}{R} \\ \frac{\partial R'}{\partial n'} &= \frac{\partial R'}{\partial z'} = \frac{z'+z}{R} \end{aligned} \right\} \frac{\partial G}{\partial n'} \Big|_S = \frac{ik}{2\pi} \frac{ze^{ikR}}{R^2} +$$

ALSO, $S_D \rightarrow 0$ SINCE $\psi=0$ ON S' . WE HAVE:

$$\psi(\vec{r}) = \frac{1}{2\pi i} \int_A \frac{e^{ikR}}{R} \left(k + \frac{i}{R} \right) \frac{z}{R} \psi_I(\vec{r}') dA'$$

$\cos \theta$

* SEE JACKSON, § 9.8. I'VE USED THE SAME NOTATION, SEE ALSO BORN & WOLF, HECHT, ETC.

Q: WHAT IS THE NEUMANN SOLUTION?

(DERIVE IF TIME PERMITS)

Discuss

- ① WHAT APPROXIMATIONS HAVE BEEN MADE IN THE RAYLEIGH-SOMMERFELD DIFFRACTION FORMULAE?
- ② JACKSON SAYS THAT THE KIRCHOFF AND RAYLEIGH-SOMMERFELD FORMULAE ARE GOOD FOR APERTURES LARGER THAN $\lambda \equiv c/\nu$. DOES THIS MAKE SENSE?

- ROUND UP / CORRECT FORMULAE
 $\psi^{(S)}$, $\psi^{(P)}$, $\psi^{(K)}$

- SHOW THAT, IN THE FAR FIELD,

$$\psi^{(S)} = \psi^{(P)} = \psi^{(K)}$$

... AND THAT THE FAR FIELD ISN'T SO FAR!

- FOR INCIDENT COLLIMATED BEAM,
 GET THE FOURIER OPTICS RESULT.

KIRCHOFF:

$$\psi_K(\vec{r}) = + \frac{1}{4\pi} \int_A dA' \left[\psi_I(\vec{r}') \frac{\partial}{\partial n'} \frac{e^{ikR}}{R} - \frac{e^{ikR}}{R} \frac{\partial \psi_I(\vec{r}')}{\partial n'} \right]$$

RAYLEIGH-SOMMERFELD:

$$\psi_{RS}^{(P)}(\vec{r}) = - \frac{1}{2\pi} \int_A dA' \frac{e^{ikR}}{R} \frac{\partial \psi_I(\vec{r}')}{\partial n'}$$

$$\psi_{RS}^{(S)}(\vec{r}) = \frac{1}{2\pi} \int_A dA' \psi_I(\vec{r}') \frac{\partial}{\partial n'} \frac{e^{ikR}}{R}$$

THESE TWO ASSUME PLANE BOUNDARY. (METHOD OF IMAGES WAS USED)

ALL OF THESE COME FROM THE THEOREM OF KIRCHOFF & HELMHOLTZ,

$$\psi(\vec{r}) = - \oint_S dA' \left[\psi(\vec{r}') \frac{\partial G}{\partial n'} - G(\vec{r}, \vec{r}') \frac{\partial \psi}{\partial n'} \right]$$

WHERE, AS USUAL, THE n' DIRECTION IS INWARD,

AND $(\nabla^2 + k^2)G = \delta(\vec{r} - \vec{r}')$, $G = \frac{-1}{4\pi|\vec{r} - \vec{r}'|} + H$, $\nabla^2 H = 0$.

NOTE: YOU'LL SEE VARIOUS VERSIONS OF THIS IN TEXTS,

OFTEN WITH - OR -4π ON RHS OF ABOVE EQUATION!

RECALL,

$$R \equiv \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\frac{\partial R}{\partial n'} = \frac{\partial R}{\partial z'} = \frac{z'-z}{R}$$

$$\frac{\partial}{\partial n'} \left(\frac{e^{ikR}}{R} \right) = -\frac{z'-z}{R^3} e^{ikR} + ik \frac{z'-z}{R^2} e^{ikR}$$

$$= \frac{e^{ikR}}{R} \left(\frac{z'-z}{R} \right) \left(ik - \frac{1}{R} \right)$$

SO,

$$\left(\frac{\partial}{\partial n'} \frac{e^{ikR}}{R} \right)_{S'} = \frac{e^{ikR}}{R} \cos \theta \left(ik - \frac{1}{R} \right)$$

SUPPOSE $\psi_{\text{I}}(\vec{r}) = \frac{e^{iks}}{s}$, $s = |\vec{r}' - \vec{r}''|$

$$\frac{\partial s}{\partial n'} = \frac{\partial s}{\partial z'} = \frac{z'-z''}{s}$$

$$\frac{\partial \psi_{\text{I}}(\vec{r}'')}{\partial n'} = \frac{e^{iks}}{s} \cos \theta' \left(ik - \frac{1}{s} \right)$$

SO

$$\psi^{(P)}(\vec{r}) = -\frac{1}{2\pi} \int_A dA' \frac{e^{ikR}}{R} \frac{e^{iks}}{s} \cos \theta' \left(ik - \frac{1}{s} \right)$$

$$\psi^{(S)}(\vec{r}) = \frac{1}{2\pi} \int_A dA' \frac{e^{ikR}}{R} \frac{e^{iks}}{s} \cos \theta \left(ik - \frac{1}{R} \right)$$