

Ph 566 In-Class Exercise: Fourier Optics

With Dirichlet boundary conditions, the Rayleigh-Sommerfeld diffraction formula is:

$$\psi_D(\mathbf{r}) = \frac{1}{2i\pi} \int_{\mathbb{A}} \frac{e^{ikR}}{R} \frac{z}{R} \left(k + \frac{i}{R} \right) \psi_I(\mathbf{r}') dA'. \quad (1)$$

where the aperture \mathbb{A} is in the plane $z' = 0$. We define

$$R \equiv \sqrt{(x - x')^2 + (y - y')^2 + z^2}. \quad (2)$$

Your task is to derive an elementary result of Fourier optics and then apply it to single slit diffraction.

1. Rewrite equation 1 by replacing $R \rightarrow z$, except in the complex exponential. Explain why this corresponds to the small angle approximation. Why would it be inappropriate to substitute $\exp(ikz)$ for $\exp(ikR)$?
2. Now take the far field limit by removing the term that falls off as z^{-2} .
3. For the complex exponential, assume that x' and y' are small compared to z . We will *not* assume that x and y are small; they may affect the phase in the complex exponential. Show that to first order,

$$R = r - \frac{xx'}{r} - \frac{yy'}{r}, \quad (3)$$

where the spherical coordinate r is defined as usual.

4. Now define new variables

$$\zeta = \frac{kx}{r}, \quad \eta = \frac{ky}{r}. \quad (4)$$

Express ψ_D in terms of the 2D Fourier transform¹ $\mathcal{F}_{x'y'}[\mathcal{A}(x', y') \psi_I(x', y', 0)]$, where $\mathcal{A} = 1$ in the aperture \mathbb{A} , and $\mathcal{A} = 0$ elsewhere.

5. Using the Fourier formulation, calculate the intensity pattern $I(x, y, z) = \psi_D^* \psi_D$ due to a plane wave $\psi_I(\mathbf{r}') = E_0 \exp(ikz')$, incident upon a single, infinitely long slit of width d . Change the Fourier coordinates (ζ, η) back to far field coordinates (x, y) .

¹Note that the variables ζ, η are angular coordinates in the far field, but they are also frequencies conjugate to x', y' .

$$\textcircled{1} \psi_0(\vec{r}) = \frac{1}{2\pi i} \int_A \frac{e^{ikR}}{z} \left(k + \frac{i}{z}\right) \psi_I(\vec{r}') dA'$$

NOTE THAT $R \rightarrow z$ DOES NOT WORK FOR THE COMPLEX EXPONENTIAL, WHICH VARIES RAPIDLY WITH ITS ARGUMENT.

\textcircled{2} FOR LARGE z ,

$$\psi_0 = \frac{k}{2\pi i z} \int_A e^{ikR} \psi_I(\vec{r}') dA'$$

\textcircled{3}

$$R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$$

FOR SMALL x' AND y' , IGNORE QUADRATIC TERMS:

$$R \approx [r^2 - 2xx' - 2yy']^{1/2}$$

$$= r \left[1 - 2[xx' + yy']/r^2\right]^{1/2} \approx r \left[1 - \frac{xx' + yy'}{r^2}\right]$$

$$\therefore R \approx r - \frac{xx'}{r} - \frac{yy'}{r} \quad \checkmark$$

$$\textcircled{4} \psi_0 = \frac{k}{2\pi i z} \iint_A e^{i(\xi x' + \eta y')} \psi_I(x', y', 0) dx' dy' \cdot e^{ikr}$$

$$= \frac{k}{2\pi i z} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' A(x', y') \psi_I(x', y', 0) e^{i(\xi x' + \eta y')} \cdot e^{ikr}$$

$$= \frac{k}{iz} \mathcal{F}_{x'y'} [A \psi_I] e^{ikr}$$

⑤ $\psi_I(x', y', 0) = E_0$

$$\begin{aligned} \psi_D &= \frac{E_0 h}{2\pi i z} \int_{-d/2}^{d/2} dx' \int_{-\infty}^{\infty} dy' e^{i(\xi x' + \eta y')} \cdot e^{i k r} \\ &= \frac{E_0 h}{i z} \int_{-d/2}^{d/2} \cos \xi x' dx' \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \eta y'} dy' \cdot e^{i k r} \\ &= \frac{E_0 h}{i z} \delta(\eta) \frac{1}{\xi} \sin(\xi x') \Big|_{x'=-d/2}^{d/2} \cdot e^{i k r} \\ &= \frac{E_0 h}{i z} \delta\left(\frac{k y}{r}\right) \frac{2}{\xi} \sin(\xi d/2) \cdot e^{i k r} \\ &= \frac{E_0 h d}{i z} \delta\left(\frac{k y}{r}\right) \frac{2}{\xi d} \sin(\xi d/2) \cdot e^{i k r} \\ &= \frac{E_0 h d}{i z} \delta\left(\frac{k y}{r}\right) \cdot e^{i k r} \cdot \text{SINC}\left(k d x / 2 r\right) \end{aligned}$$

$$I = \psi_D^* \psi_D = \frac{E_0^2 h^2 d^2}{z^2} \delta^2\left(\frac{k y}{r}\right) \text{SINC}^2(k d x / 2 r)$$

IT'S POSSIBLE TO MAKE IT A LITTLE NEATER BY FACTORING THE h OUT OF δ :

$$I = \frac{E_0^2 d^2}{z^2} \delta^2\left(\frac{y}{r}\right) \text{SINC}^2(k d x / 2 r)$$