

COMPLEX ANALYSIS (RHB CH.24)

PRESENTATION HERE FOLLOWS THE TEXT CLOSELY. RHB ITSELF GLOSSES OVER MANY MATHEMATICAL DETAILS; IF THIS BOTHERS YOU, YOU MAY WISH TO READ ALONG IN A COMPLEX ANALYSIS TEXT. I HAVE FOUND CHURCHILL & WARD* TO BE READABLE.

FUNCTIONS OF A COMPLEX VARIABLE (§24.1)

$$z = x + iy \quad (z \in \mathbb{C}; x, y \in \mathbb{R})$$

$$f(z) = u(x, y) + i v(x, y) \quad (u, v \in \mathbb{R})$$

$f(z)$ IS ANALYTIC ON DOMAIN R (SUBSET OF \mathbb{C}) IF

$$\frac{df}{dz} \equiv \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

EXISTS AND IS INDEPENDENT OF THE DIRECTION FROM WHICH Δz GOES TO ZERO, $\forall z \in R$.

* CHURCHILL, R.V. AND WARD, J.W. COMPLEX VARIABLES AND APPLICATIONS MCGRAW-HILL, 1960-1989.

EXAMPLE

SHOW THAT $f(z) = \operatorname{Re}(z)$ IS NOT ANALYTIC

$$f'(z) \equiv \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

NOW LET $\Delta z = \Delta x + i\Delta y$.

$$f' = \lim_{\Delta z \rightarrow 0} \frac{x + \Delta x - x}{\Delta x + i\Delta y} = \frac{\Delta x}{\Delta x + i\Delta y}$$

THE ANSWER DEPENDS ON OUR PATH OF APPROACH TO $\Delta z = 0$.

CASE I: $\Delta x \rightarrow 0$ FIRST ($\Delta z \rightarrow 0$ ALONG REAL AXIS)

$$\Rightarrow f' \rightarrow \frac{\pm 0}{i\Delta y} \rightarrow 0 \text{ ALONG IMAGINARY AXIS}$$

CASE II: $\Delta y \rightarrow 0$ FIRST ($\Delta z \rightarrow 0$ ALONG IMAGINARY AXIS)

$$\Rightarrow f' \rightarrow \frac{\Delta x}{\Delta x} \rightarrow 1.$$

WITH APOLOGIES TO THOSE WHO HAVE SEEN THIS, I'LL DO TWO MORE...

EXAMPLE: SHOW THAT $f(z) = 1/(z-c)$ IS ANALYTIC EVERYWHERE EXCEPT $z=c$.

$$\begin{aligned}
 f'(z) &\equiv \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
 &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\frac{1}{z+\Delta z-c} - \frac{1}{z-c} \right) \\
 &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\frac{[z-c] - [z+\Delta z-c]}{[z+\Delta z-c][z-c]} \right) \\
 &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z} \left(\frac{-\Delta z}{[z+\Delta z-c][z-c]} \right) \\
 &= \frac{-1}{(z-c)^2}, \quad \text{INDEPENDENT OF HOW } \Delta z \text{ APPROACHES ZERO.}
 \end{aligned}$$

BUT CLEARLY UNDEFINED AT $z=c$.

EXAMPLE: SHOW THAT $f(z) = |z|$ IS NOT ANALYTIC.

$$f'(z) \equiv \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \frac{|z+\Delta z| - |z|}{\Delta z}$$

OBSERVE: $\frac{\text{NUMERATOR} \in \mathbb{R}}{\text{DENOMINATOR} \in \mathbb{C}}$

SO IF $\Delta z \rightarrow 0$ ALONG REAL AXIS, $f' \in \mathbb{R}$
 IF $\Delta z \rightarrow 0$ ALONG IMAGINARY AXIS, $f' \in \mathbb{I}$

$\therefore f'$ IS UNDEFINED. QED

MORE EXAMPLES ARE IN RHB...

CAUCHY - RIEMANN RELATIONS (§ 24.2)

SUPPOSE $f(z)$ IS ANALYTIC IN SOME DOMAIN R ,
WHERE WE DEFINE

$$f' = \lim_{\Delta z \rightarrow 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right], \quad z \in R$$

LETTING $f(z) = u(x, y) + i v(x, y)$, THIS BECOMES

$$f' = \lim_{\Delta z \rightarrow 0} \left[\frac{u(x + \Delta x, y + \Delta y) - u(x, y) + i v(x + \Delta x, y + \Delta y) - i v(x, y)}{\Delta x + i \Delta y} \right]$$

I. SUPPOSE $\Delta z \rightarrow 0$ ALONG REAL AXIS, I.E. $\Delta y \equiv 0$:

$$f' = \lim_{\Delta x \rightarrow 0} \left[\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$\Rightarrow f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

II. SUPPOSE $\Delta z \rightarrow 0$ ALONG IMAGINARY AXIS, I.E. $\Delta x \equiv 0$:

$$f' = \lim_{\Delta y \rightarrow 0} \left[\frac{u(x, y + \Delta y) - u(x, y)}{i \Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i \Delta y} \right]$$

$$\Rightarrow f' = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

SINCE $f(z)$ IS SUPPOSED TO BE ANALYTIC, CASES
I & II MUST AGREE!

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$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

CAUCHY-RIEMANN
RELATIONS

- SINCE THESE FOLLOW FROM ASSUMING $f(z)$ ANALYTIC, THEY ARE A NECESSARY BUT NOT SUFFICIENT CONDITION FOR $f(z)$ TO BE ANALYTIC.
- TURNS OUT THAT EXISTENCE AND CONTINUITY OF $\partial_x u, \partial_y u, \partial_x v, \partial_y v$ PLUS THE C-R RELATIONS, ARE SUFFICIENT TO SHOW THAT $f(z)$ IS ANALYTIC.

CCK CONJECTURE: IF $f(z)$ CAN BE EXPRESSED DIRECTLY IN TERMS OF z [WITHOUT REFERENCE TO x OR y , OR ARTIFICES THAT WOULD ALLOW US TO TREAT x & y DIFFERENTLY SUCH AS $||$, $\text{Re}()$, $\text{Im}()$, $*$] THEN $f(z)$ IS PROBABLY ANALYTIC (OR CAN BE DEFINED SO THAT IT IS).

EXAMPLE

$$f(z) = |x| - i|y|$$

- IN QUADRANT I, $f(z) = x - iy = z^*$
- II, $f(z) = -x - iy = -z$
- III, $f(z) = -x + iy = -z^*$
- IV, $f(z) = x + iy = z$

CCK INTUITS: $f(z)$ ANALYTIC IN QUADRANTS II & IV ONLY.

CHECK USING CAUCHY-RIEMANN!

ANALYTIC $f(z)$ ARE THE INTERESTING KIND, AS FAR AS THE FIELD OF COMPLEX VARIABLES IS CONCERNED.

ANALYTIC = GOOD.