

CLASS BUSINESS

- NEW VERSION OF HW 12 (DUE MONDAY!) IN YOUR BOXES!

CORRECTED SIGN ERRORS: $\psi = e^{-i\omega t}$

$$(\nabla^2 + \frac{\omega^2}{c^2}) F = \delta(\vec{r} - \vec{r}_0)$$

$$(\nabla^2 + k^2) u = \rho(\vec{r})$$

- REVIEW WILL BE HELD THURS. 12/7, 4:30 - 6 PM IN THE PHYSICS LIBRARY EPS 235.

SO FAR IN CH. 24

§24.1 • ANALYTIC $f(z) = u + iv$ ($f'(z)$ WELL-DEFINED)

§24.2 • CAUCHY-RIEMANN RELATIONS

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

TODAY

§24.5 • BRANCH POINTS & BRANCH CUTS FOR MULTI-VALUED $f(z)$; $z^{1/2}$ & $\ln z$ AS EXAMPLES.

§24.2 • u & v AS SOLUTIONS TO $\nabla^2 \psi = 0$
 • STREAM FUNCTIONS (NOT IN RHB)

GOAL: BY END OF TODAY, YOU SHOULD HAVE ALL YOU NEED TO DO THE SYROVATSKII CURRENT SHEET.

UPCOMING:

§24.3-24.4 • CONVERGENCE OF COMPLEX POWER SERIES

§24.6 • SINGULARITIES & ZEROS

§§24.8 ff • INTEGRATION IN THE COMPLEX PLANE

MULTI-VALUED $f(z)$; BRANCH CUTS (§ 24.5)

CONSIDER $f(z) = z^{1/2}$:

$$\text{LET } z = r e^{i\theta}, \quad r, \theta \in \mathbb{R}$$

$$f(z) = \sqrt{r} e^{i\theta/2}$$

AFTER ONE LAP ^{OF z} AROUND THE ORIGIN, $f(z)$ DOES NOT COME BACK TO THE SAME VALUE. THUS, THE ORIGIN IS TERMED A BRANCH POINT FOR $z^{1/2}$.

Q: WHAT HAPPENS IF z TAKES 2 LAPS AROUND THE ORIGIN?

A: $z^{1/2}$ COMES BACK TO ITS ORIGINAL VALUE.

Q: HOW MANY LAPS WOULD IT TAKE FOR $z^{1/n}$ TO COME BACK TO ITS ORIGINAL VALUE?

A: $1/n$, WHERE n IS THE NUMBER OF HONEST POLITICIANS.

RETURN TO THE EXAMPLE AT HAND, $f(z) = z^{1/2}$:

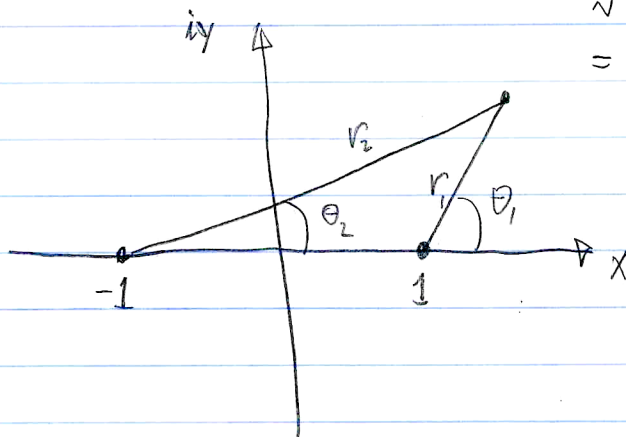
→ TO REMOVE THE MULTI-VALUED PROBLEM, WE MAY CHOOSE TO LET $0 \leq \theta < 2\pi$.

- IN THAT CASE, $z^{1/2}$ IS NO LONGER MULTIVALUED BUT
- WE HAVE MADE $z^{1/2}$ DISCONTINUOUS AT $\theta = 0$. THIS DISCONTINUITY IS CALLED A BRANCH CUT.

EXAMPLE:

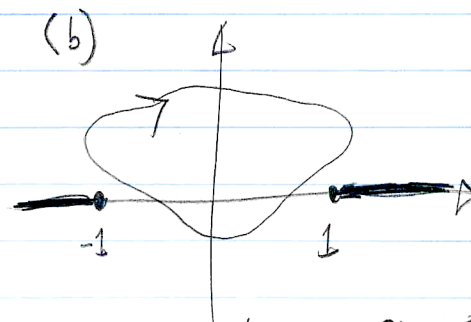
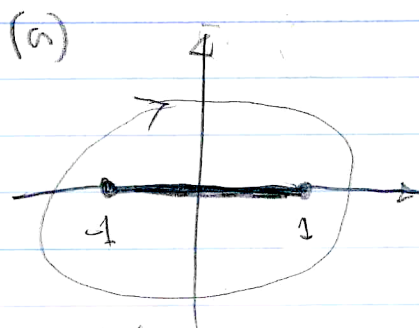
CONSIDER

$$\begin{aligned}
 f(z) &= \sqrt{z^2 - 1} \\
 &= \sqrt{z+1} \sqrt{z-1} \\
 &= \sqrt{r_1 e^{i\theta_1} r_2 e^{i\theta_2}} \\
 &= \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}
 \end{aligned}$$



ONE LAP AROUND BOTH POLES ADDS UP TO 2π , BUT 1 LAP AROUND 1 POLE IS JUST π .

BRANCH CUT POSSIBILITIES:



DISCUSS: WANT ANY CLOSED CURVE TO GO THROUGH A PHASE OF $2\pi N$, $N \in \mathbb{Z}$.

ANOTHER (VERY IMPORTANT) MULTI-VALUED FUNCTION —

$$\begin{aligned}
 f(z) &= \ln z \\
 &= \ln(re^{i\theta}) \\
 &= \ln(e^{\ln r + i\theta})
 \end{aligned}$$

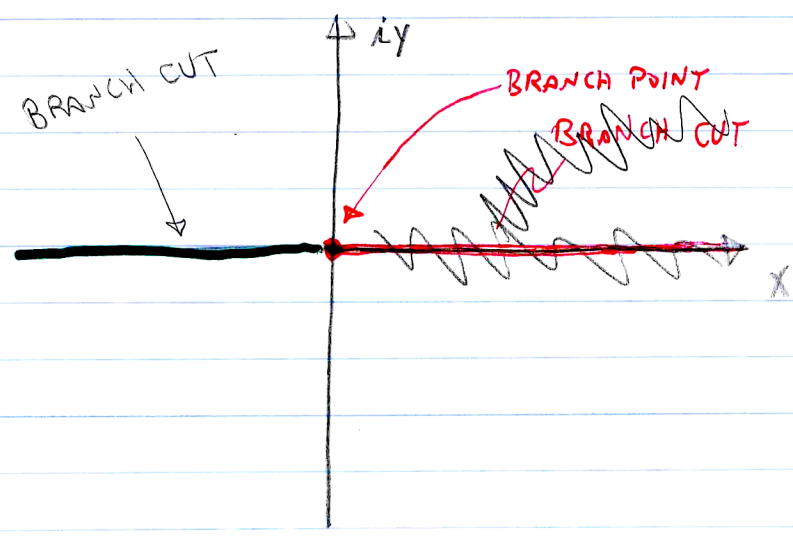
CLEARLY, THE EXPONENTIAL IS PERIODIC IN θ , SO THE RESULT IS MULTI-VALUED:

$$\ln z = \ln r + i(\theta + 2\pi n), \quad n \in \mathbb{I}$$

TRADITIONALLY, WE DEFINE

$$\ln z \equiv \ln r + i\theta, \quad -\pi < \theta < \pi$$

WHERE IS THE BRANCH POINT? THE BRANCH CUT?



BACKING UP TO §24.2 ...

RELATION OF ANALYTIC $f(z)$ TO LAPLACES EQN.:

BEGIN WITH C-R:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\downarrow \partial/\partial x$$

$$\downarrow \partial/\partial y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}$$

SIMILARLY,

$$\begin{aligned} \nabla^2 u &= 0 \\ \nabla^2 v &= 0 \end{aligned}$$

SADLY, THIS ONLY WORKS IN 2D.

ANOTHER FABULOUS FACT:

$$\begin{aligned} \nabla u \cdot \nabla v &= \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \\ &= \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} = 0. \end{aligned}$$

\therefore $\nabla u \perp \nabla v$

CONTOURS OF $u(x,y) \perp$ CONTOURS OF $v(x,y)$

AND

CONTOURS OF $v \parallel \nabla u$
 CONTOURS OF $u \parallel \nabla v$

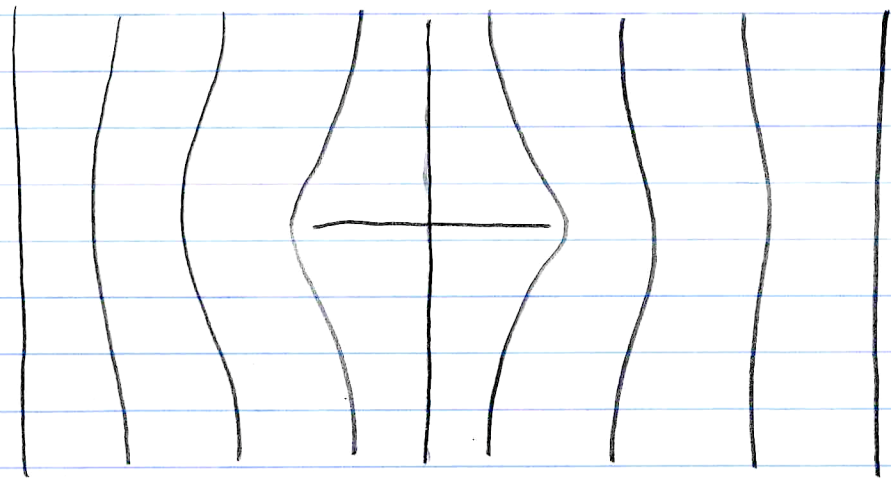
EXAMPLE:

$$f(z) = \sqrt{z^2 - 1} = \sqrt{(z-1)(z+1)}$$

Q: IS IT ANALYTIC?

A: YES — BUT IF YOU TAKE $\partial/\partial z$, YOU'LL SEE
POLES AT $z = \pm 1$.

CONSIDER — $\nabla^2 u = 0$, $u \equiv \text{Re}(f(z))$.

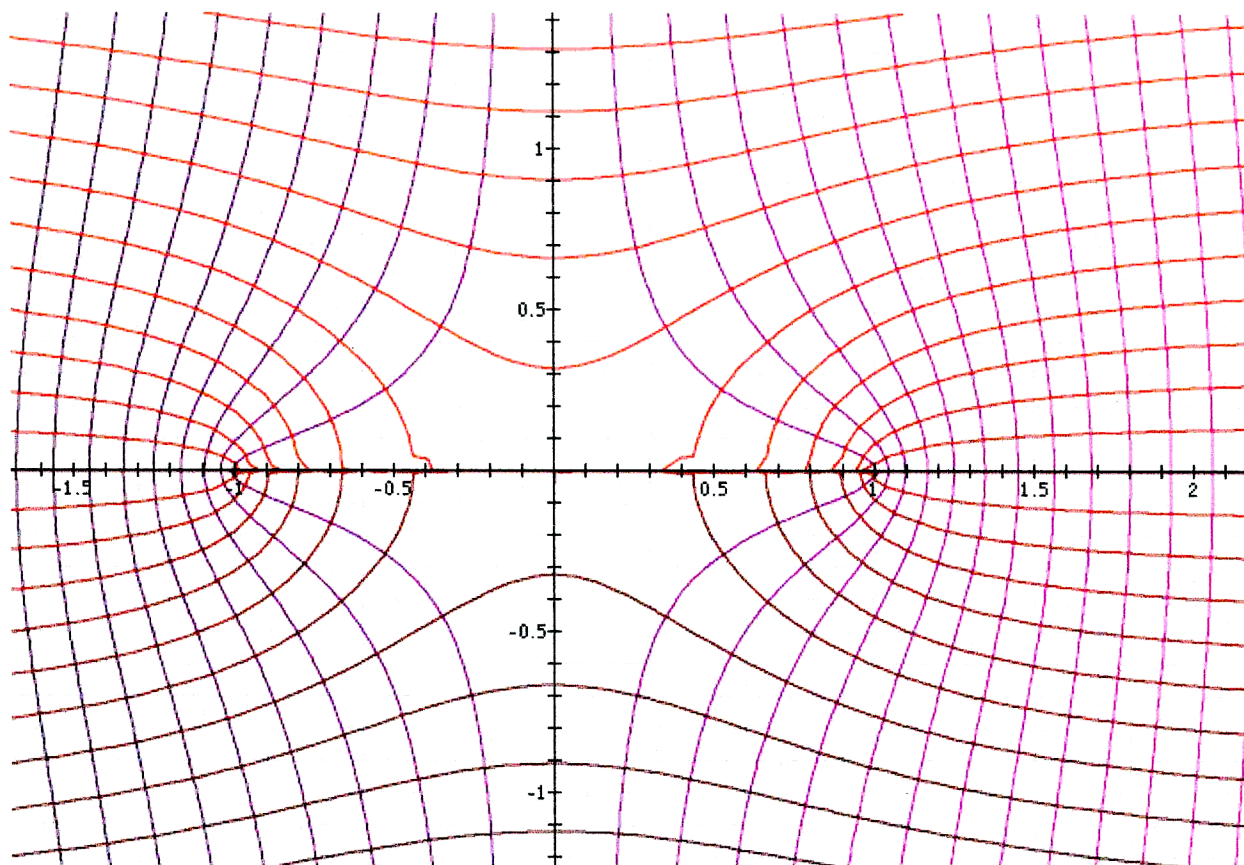


PHYSICALLY: CONDUCTING PLATE IMMERSSED IN
A HORIZONTAL, UNIFORM \vec{E} .

SINCE CONTOURS OF $v \parallel \nabla u$, THEY MARK
THE FIELD LINES.

■ $\text{Re} ((x + iy - 1)^{1/2} (x + iy + 1)^{1/2})$

■ $\text{Im} ((x + iy - 1)^{1/2} (x + iy + 1)^{1/2})$



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Does $\vec{E} \rightarrow E_0 \hat{x}$, i.e. $u \rightarrow -E_0 x$, AS $z \rightarrow \infty$?

$$u = \operatorname{Re} \left[(x+iy+1)(x+iy-1) \right]^{1/2}$$

$$\text{AS } z \rightarrow \infty, u \rightarrow \operatorname{Re} \left[(z)(z) \right]^{1/2} = \operatorname{Re} z = x \quad \checkmark$$

THERE IS A WHOLE COTTAGE INDUSTRY IN FINDING SIMPLE MODELS OF COMPLICATED SOLUTIONS TO $\nabla^2 \phi = 0$ THIS WAY. THESE HAVE BEEN EXPLOITED FOR MODELING SOME SYSTEMS WITH VERY INTERESTING BOUNDARY CONDITIONS.

MORE ON THAT LATER!

STREAM FUNCTIONS

CONSIDER SOME ANALYTIC FUNCTION $f(z) = u(x,y) + iv(x,y)$

CAUCHY-RIEMANN:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

SUPPOSE WE USE u AS A SCALAR POTENTIAL TO MODEL $\nabla \cdot \vec{E} = 0$,
I.E.

$$\vec{E} = -\nabla u, \quad \nabla^2 u = 0.$$

WHAT IS THE SIGNIFICANCE OF v ?

$$\nabla v \cdot \nabla u = 0,$$

SO CONTOURS OF v ARE FIELD LINES. MOREOVER,

$$\left. \begin{aligned} E_x &= -\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \\ E_y &= -\frac{\partial u}{\partial y} = +\frac{\partial v}{\partial x} \end{aligned} \right\} \text{(CAUCHY-RIEMANN)}$$

SO $v(x,y)$ CAN BE USED MUCH LIKE A SCALAR POTENTIAL TO GET THE VECTOR FIELD $\vec{E}(x,y)$.

$v(x,y)$ IS CALLED A STREAM FUNCTION. (USED A LOT IN FLUID MECHANICS)

→ IN GENERAL, ANY $\Phi(x,y) \in \mathbb{R}$ SATISFYING $\nabla^2 \Phi = 0$ CAN BE USED AS A STREAM FUNCTION FOR A SOLENOIDAL VECTOR FIELD \vec{B} (THAT IS, \vec{B} SATISFYING $\nabla \cdot \vec{B} = 0$):

$$\text{LET } \vec{B} = -\frac{\partial \Phi}{\partial y} \hat{x} + \frac{\partial \Phi}{\partial x} \hat{y},$$

$$\nabla \cdot \vec{B} = -\frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial x \partial y} = 0. \quad \checkmark$$

→ APPLY TO PREVIOUS EXAMPLE...