

AIRFOIL THEORY (2D)



WHAT EQUATION & B.C.'S CAN WE USE TO DESCRIBE $\vec{v}(\vec{r})$ IN THE NEIGHBORHOOD OF THE AIRFOIL? HOW CAN WE CALCULATE LIFT?

OUTLINE

- POTENTIAL FLOW
- FLOW AROUND A CYLINDER
- JOUKOWSKY AIRFOILS
- KUTTA-JOUKOWSKY THEOREM (LIFT)

NAVIER-STOKES: POTENTIAL IRRATIONAL FLOW

$$\rho \frac{D\vec{v}}{Dt} = \rho \vec{g} - \nabla p - \mu \nabla^2 \vec{v}$$

~~STEADY: $\frac{\partial}{\partial t}$~~

GRAVITATIONAL POTENTIAL

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\rho \nabla \Phi - \nabla p - \mu \nabla^2 \vec{v} \quad \star$$

~~STEADY: $\frac{\partial}{\partial t}$~~

INVISCID: $\mu \rightarrow 0$

~~$(\vec{v} \cdot \nabla) \vec{v} + \rho \nabla \Phi + \nabla p = 0$~~

NOW CONSIDER THE EVOLUTION OF THE VORTICITY, $\vec{\omega} \equiv \nabla \times \vec{v}$.
TAKE THE CURL OF \star :

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times [(\vec{v} \cdot \nabla) \vec{v}] = 0$$

IDENTITY: $(\vec{v} \cdot \nabla) \vec{v} = \frac{1}{2} \nabla v^2 - \vec{v} \times \vec{\omega}$, SO

$$\frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega})$$

IF INITIALLY, $\vec{\omega} = 0$, THEN $\vec{\omega} = 0$ FOREVER. (MOREOVER, VISCOSITY TENDS TO REDUCE $\vec{\omega}$!) SO LET'S ASSUME $\nabla \times \vec{v} = 0$.

FURTHER, LET US ASSUME INCOMPRESSIBLE FLOW: $\nabla \cdot \vec{v} = 0$, $\rho = \text{CONST}$ (EQUIVALENTLY, THE FLOW IS SUBSONIC, $V_{\infty} \ll C_s$).

EQUATION * IS NOW:

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla v^2 \right] = -\rho \nabla \Phi - \nabla p$$

FOR OUR AIRFOIL, WE SEEK A STEADY FLOW SOLUTION, $\frac{\partial}{\partial t} = 0$:

$$\nabla \left(\frac{1}{2} \rho v^2 + \rho \Phi + p \right) = 0$$

SO, THE QUANTITY $H = \frac{1}{2} \rho v^2 + \rho \Phi + p$ IS
 A GLOBAL CONSTANT OF THE FLOW (COMPARE BERNOULLI'S LAW).
 THIS ALLOWS US TO ~~can~~ CALCULATE PRESSURE EVERYWHERE,
 GIVEN $\vec{v}(\vec{r})$. \vec{v} ITSELF MUST SATISFY

$$\nabla \times \vec{v} = 0, \quad \nabla \cdot \vec{v} = 0.$$

GENERAL SOLUTION:

$$\vec{v} = \nabla \phi, \quad \underline{\underline{\nabla^2 \phi = 0}}$$

WE CALL THIS A "POTENTIAL FLOW."

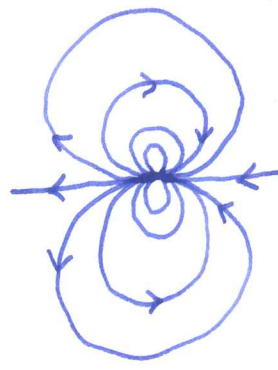
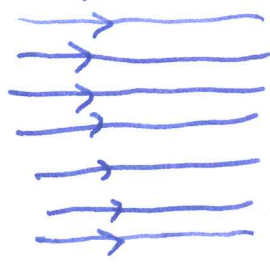
FLOW AROUND A CYLINDER

RECALL THAT $f(z)$ IS A SOLUTION TO $\nabla^2 f = 0$.

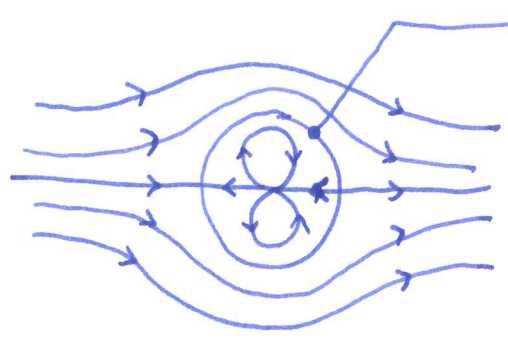
LET $f_1(z) = V_{\infty} \left(z + \frac{1}{z} \right)$, $\vec{v}_1 \equiv \nabla \text{Re}(f_1(z))$

CONST.
 UNIFORM VELOCITY OF $V_{\infty} \hat{x}$
 SOURCE DOUBLET (2D DIPOLE)

CONTOURS OF $\text{Im}(f_1)$ ARE STREAMLINES.

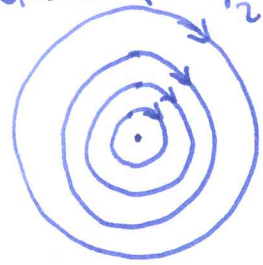


ADDED TOGETHER:



UNIT CIRCLE IS A STREAMLINE.
 $\vec{v}_1 \rightarrow V_{\infty} \hat{x}$ AS $r \rightarrow \infty$.

NOW CONSIDER $f_2(z) = \frac{i\Gamma \ln z}{2\pi}$, $\vec{v}_2 \equiv \nabla \text{Re}(f_2(z))$

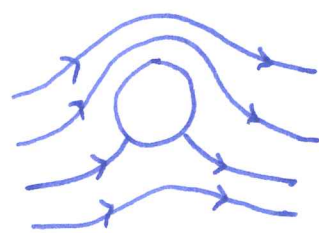


CONST.
 $\vec{v}_2 = \frac{-d}{dr} \frac{i\Gamma \ln r}{2\pi} \hat{\phi} = -\hat{\phi} \frac{\Gamma}{2\pi r}$

$\oint_{r=R} \vec{v}_2 \cdot d\vec{\ell} = \frac{-\Gamma}{2\pi R} \cdot 2\pi R = -\Gamma \equiv \text{CIRCULATION}$

NOTE THAT $\nabla \times \vec{v}_2 = -\Gamma \delta(\vec{r})$.
 (ZERO EXCEPT AT ORIGIN)

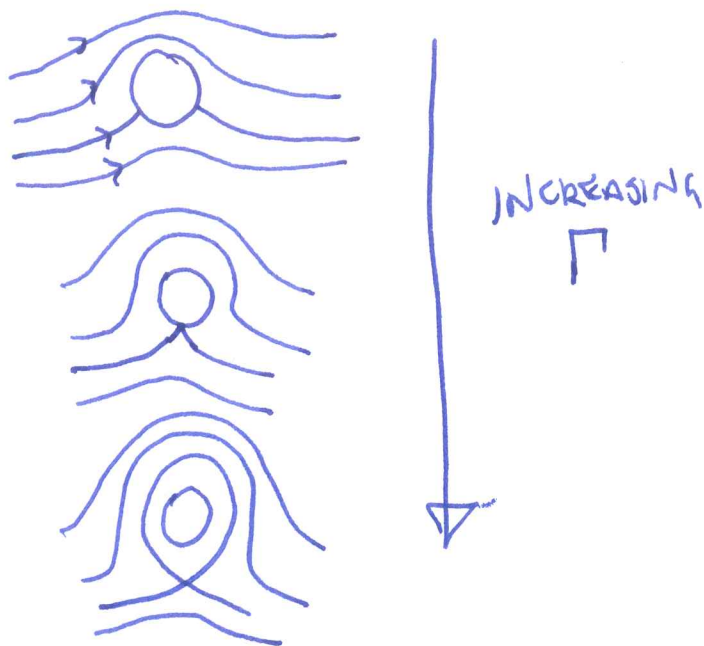
LET $F(z) = f_1 + f_2$:



(DISCUSS HOW APPEARANCE CHANGES WITH INCREASING Γ ; AND SAILING APPLICATIONS!)

NOW LET $F(z) = f_1 + f_2$:

IN PRACTICE,
THE CIRCULATION
IS INTRODUCED
BY SPINNING
THE CYLINDER.



THERE IS AN INCREASING UPWARD "MAGNUS" FORCE WITH INCREASING CIRCULATION.

- ANTON FLETTNER PATENTED THE "ROTOR SHIP" IN 1922.
- COUSTEAU'S SHIP ALCYONE - 1985.
(HIS "TURBOSAIL" IS A LITTLE MORE COMPLICATED THAN FLETTNER'S DESIGN)

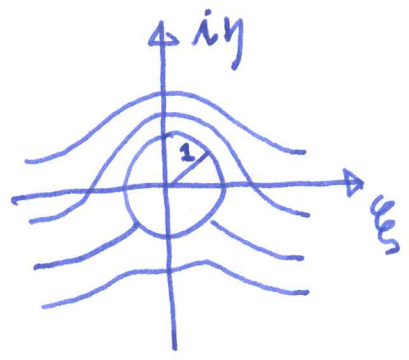
→ HOW MIGHT WE CALCULATE THE MAGNUS FORCE?

→ WHAT IS THE ELECTROSTATIC ANALOGY?

JOUKOWSKY TRANSFORM (AN EXAMPLE OF CONFORMAL MAPPING)

ξ-PLANE

$$\tilde{F}(\xi) = f_1(\xi) + f_2(\xi)$$

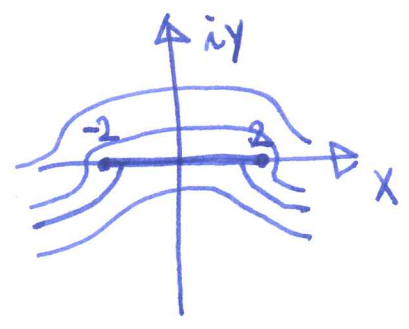


$$z = \xi + \frac{1}{\xi}$$

→

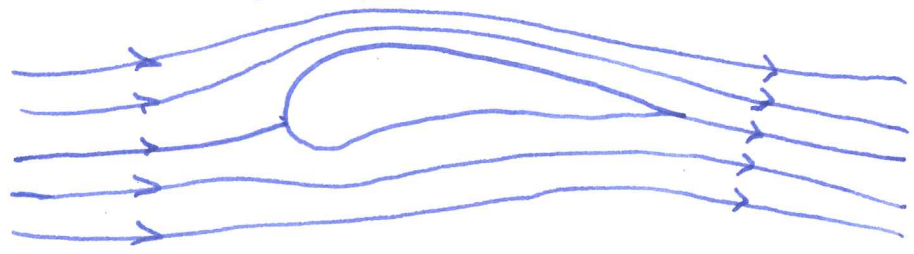
z-PLANE

$$F(z) = \tilde{F}(\xi(z))$$

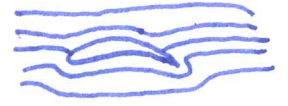


NOW HAVE A LOOK AT THE EXAMPLES ON THE WEB..

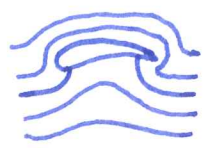
CORRECT Γ :



Γ TOO SMALL:



Γ TOO LARGE:



KUTTA-JOUKOWSKY CONDITION:

Γ SHOULD BE CHOSEN TO PUT THE TRAILING STAGNATION POINT ON THE CUSP OF THE AIRFOIL.

- THIS AVOIDS $|\vec{v}| \rightarrow \infty$
- MOMENTUM DOMINATES OVER VISCOSITY (HIGH REYNOLDS NUMBER)
- THIS IS WHAT HAPPENS NATURALLY WITH A SHARP TRAILING EDGE!

NOTE THAT THE BOUNDARY CONDITIONS ARE ~~ON~~
 "INFINITE SLIP" — POTENTIAL $\nabla \Phi \cdot \hat{n} = 0$, OR
 STREAM FN. $\nabla \Psi \times \hat{n} = 0$

Q: HOW TO CALCULATE LIFT?

A: ~~THE~~ KUTJA-JOUKOWSKY THEOREM

$$F_L = \rho V_\infty \Gamma L$$

↑ ↑ ↑ ←
 FLUID DENSITY WIND SPEED CIRCULATION WING LENGTH

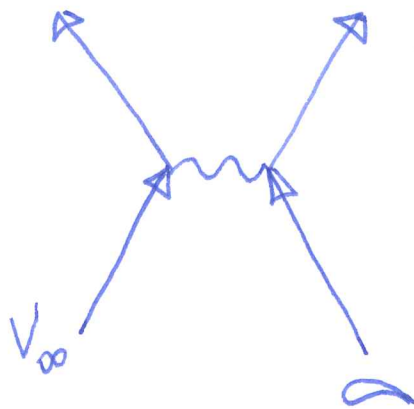
Q: IS THIS PLAUSIBLE? DOES WING SHAPE COME INTO IT?!

A: WING SHAPE DETERMINES Γ VIA K-J CONDITION.

THE TRADITIONAL DERIVATION IS A BIT TEDIOUS. USES INTEGRATION OF PRESSURE FORCE ON AIRFOIL IN COMPLEX PLANE. REQUIRES RESIDUE THEOREM, AND CAREFUL DISSECTION OF A LAURENT SERIES.

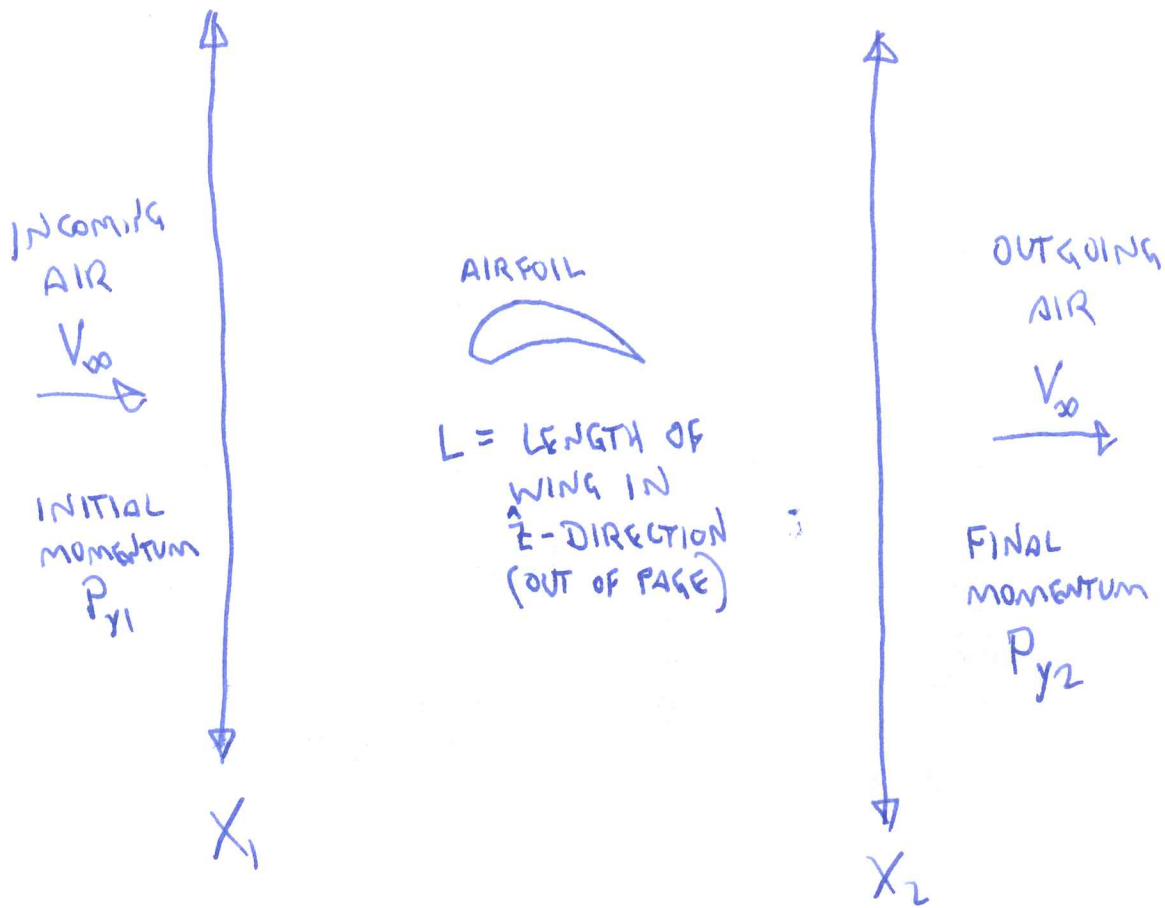
(WHEN YOU HAVE A HAMMER IN YOUR HAND, LOTS OF THINGS START TO LOOK LIKE NAILS!)

SIMPLER DERIVATION — FIND THE RATE AT WHICH THE AIRFOIL IMPARTS MOMENTUM TO THE FLUID.



IN OTHER WORDS, IT'S A SCATTERING PROBLEM...

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Q: IF $\vec{v} \rightarrow V_\infty \hat{x}$ FAR FROM AIRFOIL FOR $x \rightarrow \pm \infty$,
 HOW CAN IT BE THAT MOMENTUM WAS IMPARTED TO THE AIR?
 A: THE SAME WAY $\oint \vec{E} \cdot d\vec{A} = Q_{ENC}/\epsilon_0$ AS $r \rightarrow \infty$.

y-MOMENTUM PER VOLUME: ρv_y

y-MOMENTUM PER TIME, PER AREA: $V_\infty \rho v_y$

INCOMING y-MOMENTUM PER TIME: $\iint_A dA \rho V_\infty v_y(x_1)$

$$= \int_{-\infty}^{\infty} L \rho V_\infty v_y(x_1, y) dy$$

OUTGOING y-MOMENTUM PER TIME: $L \rho V_\infty \int_{-\infty}^{\infty} v_y(x_2, y) dy$


$$-\Delta P \text{ PER TIME} = \text{LIFT} = L \rho V_\infty \int_{-\infty}^{\infty} (v_{y1} - v_{y2}) dy$$

$$= L \rho V_\infty \Gamma \quad \underline{\underline{QED}}$$

HOW TO MODEL A WING

- PICK A JOUKOWSKY PROFILE OF DESIRED SHAPE
- SET ANGLE OF ATTACK BY ROTATING THE z -TERM IN THE CYLINDER FLOW:

$$f_1 = V_\infty \left(z e^{i\alpha} + \frac{1}{z e^{i\alpha}} \right)$$

- SET Γ USING KUTTA-JOUKOWSKY CONDITION 
- CALCULATE LIFT USING K-J THEOREM, $F_L = L \rho V_\infty \Gamma$.

Q: IS Γ INVARIANT UNDER JOUKOWSKY TRANSFORM?

A: Γ IS INVARIANT UNDER CONFORMAL MAPPING:

$$\text{LET } w(z) = \frac{dF(z)}{dz}, \quad \tilde{w}(\xi) = w(z) \frac{dz}{d\xi} = \frac{d\tilde{F}(\xi)}{d\xi}$$

$$\Gamma = \text{Re} \left[\oint w(z) dz \right] \quad (\text{MIGHT WANT TO SHOW THIS!})$$

$$\oint w(z) dz = \oint \tilde{w}(\xi) \frac{d\xi}{dz} dz = \oint \tilde{w}(\xi) d\xi$$

$$\therefore \Gamma = \tilde{\Gamma} \quad \underline{\text{QED}}$$