



## §24.3 — CONVERGENCE OF COMPLEX POWER SERIES

①

SUPPOSE WE HAVE

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C} \\ &= \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \end{aligned}$$

THIS SERIES WILL CERTAINLY CONVERGE IF THE FOLLOWING CONVERGES:

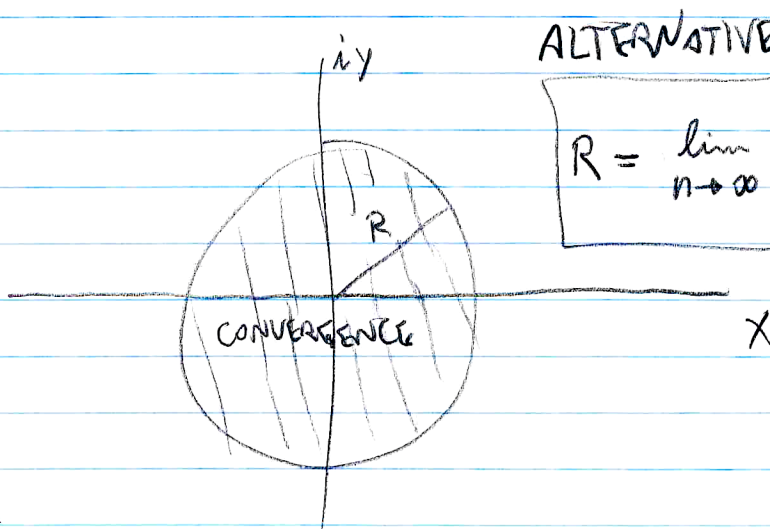
$$\sum_{n=0}^{\infty} |a_n| r^n$$

THE CONVERGENCE OF THIS REAL SERIES CAN BE EVALUATED USING THE CAUCHY ROOT TEST: (§4.5.1)

$$\begin{aligned} &\text{THE SERIES CONVERGES FOR } |z| < R, \\ &\text{WHERE } R = \lim_{n \rightarrow \infty} |a_n|^{-1/n} \end{aligned}$$

$R$  IS CALLED THE RADIUS OF CONVERGENCE.

(FOR  $|z| = R$ , YOU JUST HAVE TO LOOK CAREFULLY)



EXAMPLE — FIND THE RADIUS OF CONVERGENCE:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$|a_n| = \frac{1}{n!}$$

$$R = \lim_{n \rightarrow \infty} |a_n|^{-1/n} = \lim_{n \rightarrow \infty} |n!|^{1/n}$$

FOR LARGE  $n$ ,  $n! \rightarrow \sqrt{2\pi n} n^n e^{-n}$

SO

$$R = \lim_{n \rightarrow \infty} \left[ \sqrt{2\pi n} n^n e^{-n} \right]^{1/n}$$

$$= \lim_{n \rightarrow \infty} \sqrt{2\pi} n^{1/2n} n^1 e^{-1}$$

$\rightarrow \infty$

MUCH  
EASIER  
WITH  
THE  
RATIO  
TEST

SO THE SERIES CONVERGES FOR ALL  $z$ .

MORE EXAMPLES IN RHB:

$$\sum_{n=0}^{\infty} n! z^n \rightarrow R=0$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n} \rightarrow R=1$$

## § 24.6 SINGULARITIES & ZEROS OF $f(z)$

SINGULARITY: ANY POINT WHERE  $f(z)$  IS NOT ANALYTIC.

IF THE SINGULARITY IS CONFINED TO A POINT, IT IS CALLED AN ISOLATED SINGULARITY.

Q: FOR  $\ln z$ , THE ORIGIN IS A SINGULARITY. IS IT AN ISOLATED SINGULARITY?

A: NO. IT IS A BRANCH POINT, AND  $\ln z$  IS ALSO NOT ANALYTIC ALONG THE BRANCH CUT.

$\therefore$  BRANCH POINTS ARE NOT ISOLATED SINGULARITIES.

A POLE OF ORDER  $n$  IS AN <sup>ISOLATED</sup> SINGULARITY OF THE FORM

$$f(z) = \frac{g(z)}{(z-z_0)^n} \quad (24.23)$$

MORE PRECISELY, WE MEAN THAT

$$\lim_{z \rightarrow z_0} [(z-z_0)^n f(z)] = g(z_0), \text{ WHERE}$$

$g(z_0) \in \mathbb{C}$  IS FINITE,  $\neq 0$

- A POLE OF ORDER  $n=1$  IS A SIMPLE POLE
- IF  $g(z_0) = 0$ , WE HAVE A POLE OF ORDER  $< n$

OR  $f(z)$  IS ANALYTIC AT  $z_0$ .

- IF  $g(z_0) \rightarrow \infty$ , IT IS A POLE OF ORDER  $> n$
- IF THE LIMIT IS  $\infty$  REGARDLESS OF  $n$ , THE SINGULARITY IS ESSENTIAL.

FINAL TYPE OF SINGULARITY:

$$f(z) \rightarrow 0 \text{ OR } \infty \text{ AS } z \rightarrow z_0$$

BUT  $\lim_{z \rightarrow z_0} f(z)$  EXISTS AND IS WELL-DEFINED,

THEN THE SINGULARITY IS "REMOVABLE".

EXAMPLE

IDENTIFY &amp; CLASSIFY THE SINGULARITIES:

$$(1) \frac{1}{1-z} - \frac{1}{1+z} = f(z)$$

$$(2) f(z) = \tanh(z) = \frac{\sinh z}{\cosh z} = \frac{(e^z - e^{-z})/2}{(e^z + e^{-z})/2}$$

$$(3) f(z) = \frac{\sin z}{z}$$

(1) POLE AT  $z=1$ :

$$\lim_{z \rightarrow 1} [f(z)(z-1)] = \lim_{z \rightarrow 1} \left[ -1 - \frac{z-1}{1+z} \right] = -1$$

 $\Rightarrow$  SIMPLE POLE AT  $z=1$ (SIMILARLY AT  $z=-1$ )

$$(2) \tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\text{SINGULARITIES SATISFY } e^z = -e^{-z} \\ = e^{i(2m+1)\pi} e^{-z}, \quad m \in \mathbb{Z}$$

$$\Rightarrow z = i\pi \left( m + \frac{1}{2} \right)$$

TRY  $m=1$ !

$$\lim_{z \rightarrow i\pi(m+\frac{1}{2})} \left[ \frac{[z - i\pi(m+\frac{1}{2})]^1 \sinh z}{\cosh z} \right] = ?$$

SIMPLE POLE

USE L'HOPITAL'S RULE

$$= \lim_{z \rightarrow i\pi(m+\frac{1}{2})} \left[ \frac{[z - i\pi(m+\frac{1}{2})] \cosh z + \sinh z}{\sinh z} \right] = 1$$

$$(3) \quad f(z) = \frac{\sin z}{z}$$

$$f(z) = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

THE SINGULARITY IS REMOVABLE.

ZEROS OF  $f(z)$  ARE CLASSED LIKE POLES.

IF  $f(z) = (z - z_0)^n g(z)$ ,  $g(z)$  FINITE AT  $z = z_0$ ,

THEN  $f(z)$  HAS A ZERO OF ORDER  $n$  AT  $z_0$ .