

SO FAR IN CH. 24

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- 24.2 CAUCHY-RIEHMANN
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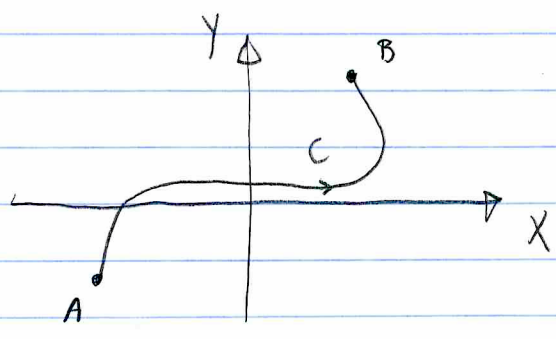
TODAY

- § 24.8 COMPLEX INTEGRALS
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UPCOMING

- § 24.10 CAUCHY'S INTEGRAL FORMULA
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§24.8 - COMPLEX INTEGRALS



CLEARLY, $\int_A^B f(z) dz$ IS AMBIGUOUS - THERE ARE AN INFINITE NUMBER OF PATHS CONNECTING THESE ENDPOINTS.

WE MAY SPECIFY A ^{PARAMETERIZED} PATH C AS FOLLOWS:

$$\left. \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned} \right\} \begin{aligned} A &= x(\alpha) + iy(\alpha) \\ B &= x(\beta) + iy(\beta) \end{aligned}$$

AND THEN REWRITE THE INTEGRAL:

$$\int_C f(z) dz = \int_C [u(z) + iv(z)] (dx + i dy)$$

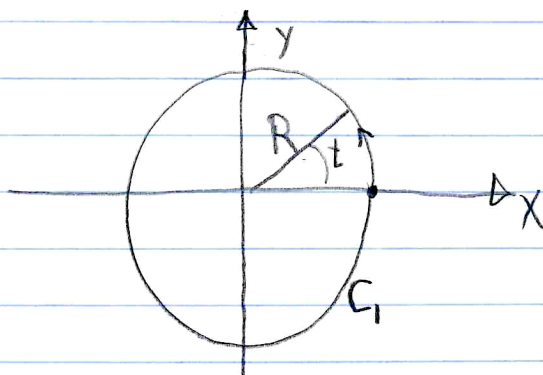
WHERE $dx = \dot{x} dt$ AND

$dy = \dot{y} dt$

$$\int_C f(z) dz = \int_{\alpha}^{\beta} u \dot{x} dt - \int_{\alpha}^{\beta} v \dot{y} dt + i \int_{\alpha}^{\beta} u \dot{y} dt + i \int_{\alpha}^{\beta} v \dot{x} dt$$

EXAMPLE:

$$f(z) = \frac{1}{z}$$



EVALUATE: $\int_{C_1} f(z) dz.$

$$f(z) = \frac{1}{z}$$

$$z(t) = R e^{it}$$

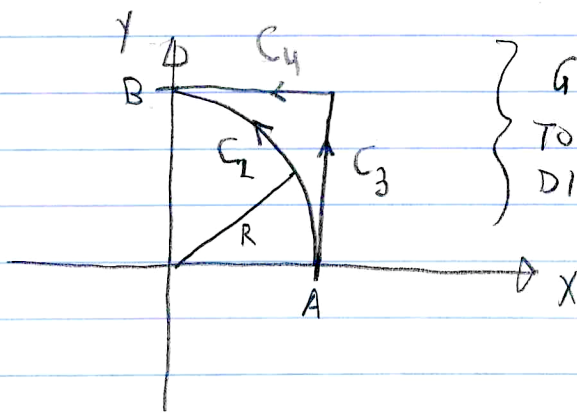
$$dz = i R e^{it} dt = i \overset{z}{z} dt$$

$$\begin{aligned} \int_{C_1} f(z) dz &= \int_0^{2\pi} f(z(t)) \dot{z} dt \\ &= \int_0^{2\pi} i dt = \underline{2\pi i}. \end{aligned}$$

INTERESTING — THE RESULT IS INDEPENDENT OF R!

EXAMPLE:

$$f(z) = \frac{1}{z}$$



(a) EVALUATE $\int_{C_2} f(z) dz$:

FROM OUR PREVIOUS EXAMPLE, THIS IS JUST

$$\int_0^{\pi/2} i dt = \frac{\pi}{2} i.$$

(b) EVALUATE $\int_{C_3} f(z) dz$:

$$\text{LET } z(t) = R(1+it), \quad 0 \leq t \leq 1.$$

$$dz = iR dt$$

now,

$$\int_{C_3} f(z) dz = \int_0^1 \frac{iR dt}{R(1+it)} = i \int_0^1 \frac{dt}{1+it}$$

$$= \ln(1+it) \Big|_0^1 = \ln(1+i) - \ln(1)$$

$$= \ln(1+i) = \ln(\sqrt{2} e^{i\pi/4})$$

$$= \frac{1}{2} \ln 2 + i\pi/4$$

CONT.



(c) EVALUATE $\int_{C_4} f(z) dz$:

$$\text{LET } z(t) = R(i+1-t),$$

$$dz = -R dt$$

$$\begin{aligned} \int_{C_4} f(z) dz &= \int_0^1 \frac{-R dt}{R(i+1-t)} = - \int_0^1 \frac{dt}{i+1-t} \\ &= \ln(i+1-t) \Big|_0^1 = \ln(i) - \ln(i+1) \\ &= \ln(e^{i\pi/2}) - \ln(\sqrt{2} e^{i\pi/4}) \\ &= i\frac{\pi}{2} - \left[\frac{1}{2} \ln 2 + i\frac{\pi}{4} \right] \\ &= i\frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$

(d) NOW, NOTE THAT:

$$\begin{aligned} \int_{C_3+C_4} f(z) dz &= \left[\int_{C_3} dz + \int_{C_4} dz \right] f(z) = i\frac{\pi}{2} \\ &= \int_{C_2} f(z) dz \end{aligned}$$

THE PATH FROM A TO B DID NOT MATTER!

ASIDE: GREEN'S THEOREM IN A PLANE (§11.3)

YOU KNOW THE DIVERGENCE THEOREM,

$$\oint_S \vec{B} \cdot d\vec{A} = \int_V \nabla \cdot \vec{B} \, dV$$

IN 2D, THIS BECOMES:

$$\oint_C \vec{B} \cdot \hat{n} \, dl = \int_A \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} \right) dx dy$$

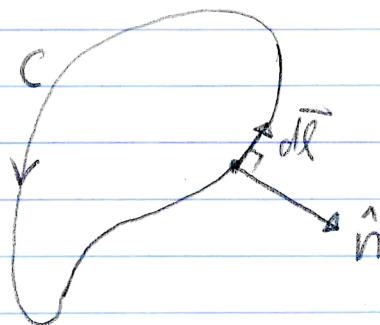
WE MAY IN GENERAL LET

$$B_x = f(x, y) \text{ AND } B_y = g(x, y).$$

ALSO, $d\vec{l} = \hat{x} dx + \hat{y} dy$

$$\text{AND } \hat{n} = \frac{\hat{x} dy - \hat{y} dx}{dl}$$

(ASSUMES CCW INTEGRATION)*



SO THE DIVERGENCE THEOREM MAY BE WRITTEN

$$\oint_C [f dy - g dx] = \int_A \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right] dx dy$$

SIGN DIFFERENCE REFLECTS CCW DIRECTION OF INTEGRATION.

* THIS BEARS THINKING ABOUT!

§ 24.9 — CAUCHY'S THEOREM

CONSIDER THE INTEGRAL AROUND A CLOSED
CONTOUR,

$$\oint_C f(z) dz = \oint_C (-v dy + u dx) + i \oint_C (u dy + v dx)$$

APPLY GREEN'S THEOREM TO BOTH INTEGRALS:

$$\oint_C f(z) dz = \int_A \left[\frac{\partial(-v)}{\partial x} + \frac{\partial(u)}{\partial y} \right] dx dy + i \int_A \left[\frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} \right] dx dy$$

SUPPOSE WE NOW USE THE CAUCHY - RIEMANN
RELATIONS:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

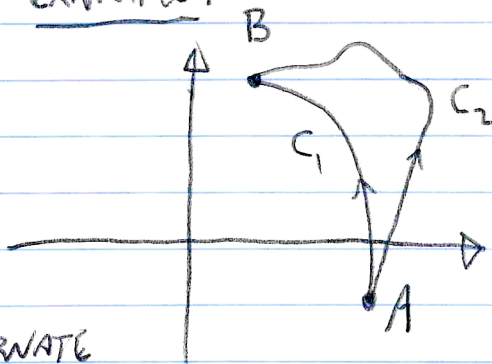
THEN, $\boxed{\oint_C f(z) dz = 0}$

Q: WHAT HAVE WE ASSUMED ABOUT $f(z)$?

A: THAT IT IS ANALYTIC EVERYWHERE INSIDE AND ON C.

RETURNING TO OUR EXAMPLE:

$$f(z) = \frac{1}{z}$$



C_1, C_2 ARE ALTERNATE
PATHS FROM A TO B.

Q: UNDER WHAT CONDITIONS DOES

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz ?$$

CONSIDER

$$\oint_{C_1 - C_2} f(z) dz = \int_{C_1} f(z) dz - \int_{C_2} f(z) dz$$

OR CROSS

IF PATH $C_1 - C_2$ DOES NOT ENCLOSE THE ORIGIN,
THEN $f(z)$ IS ANALYTIC INSIDE AND ON $C_1 - C_2$.
THEN

$$\int_{C_1} f dz - \int_{C_2} f dz = 0$$

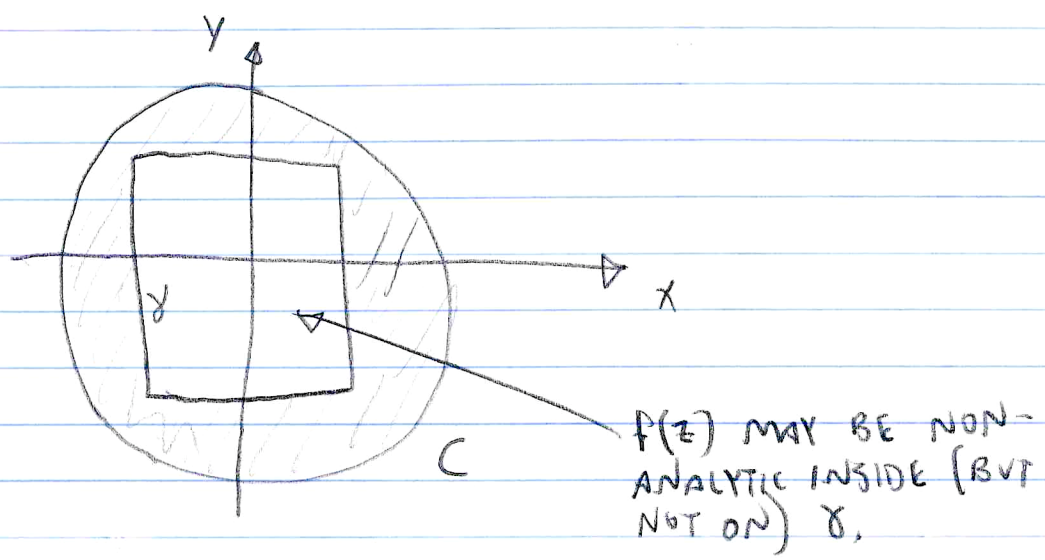
$$\therefore \int_{C_1} f dz = \int_{C_2} f dz.$$

THIS EXPLAINS A LOT. BUT DOES IT EXPLAIN
WHY OUR FIRST RESULT WAS INDEPENDENT OF R?

CONSIDER ...

A CONSEQUENCE OF CAUCHY'S THEOREM...

SUPPOSE THAT THE CLOSED CURVE γ LIES COMPLETELY INSIDE THE CLOSED CURVE C :

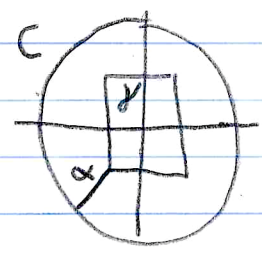


IF $f(z)$ IS ANALYTIC IN THE AREA BETWEEN THE CURVES, THEN

$$\oint_C f(z) dz = \oint_\gamma f(z) dz$$

PROOF: USE A PATH α CONNECTING γ TO C TO MAKE A CLOSED CONTOUR; THEN CAUCHY'S THEOREM SAYS:

$$\oint_C f dz + \int_\alpha f dz - \oint_\gamma f dz - \int_\alpha f dz = 0$$

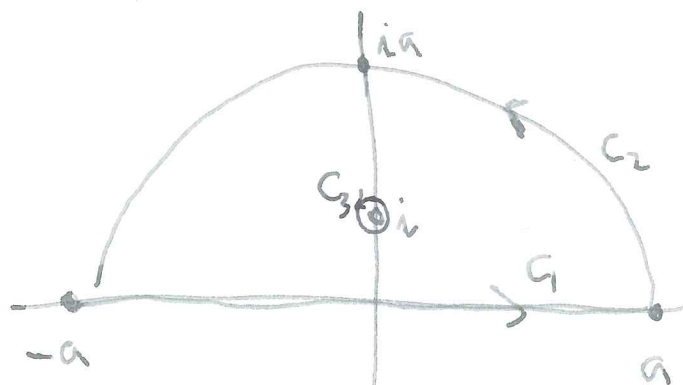


$$\Rightarrow \oint_C f dz = \oint_\gamma f dz \quad \checkmark$$

SO YES, CAUCHY'S THEOREM DOES EXPLAIN WHY OUR FIRST RESULT WAS INDEPENDENT OF R . (IT IS ALSO INDEPENDENT OF THE SHAPE OF THE CURVE ENCLOSED THE POLE!)

EXAMPLE

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$



$$\oint_{C_1, C_2} f(z) dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + \int_0^{2\pi} \frac{iz d\theta}{(z+i)(z-i)}$$

FOR LARGE z ?

ON C_3 :

$$\text{LET } z = i + \Delta z = i + \epsilon e^{i\alpha}$$
$$dz = i\epsilon e^{i\alpha} d\alpha$$

$$\int_{C_3} f(z) dz = \int_0^{2\pi} \frac{i\epsilon e^{i\alpha} d\alpha}{(2i + \epsilon e^{i\alpha}) \epsilon e^{i\alpha}} \approx \int_0^{2\pi} \frac{i d\alpha}{2i} = \pi$$

BY CAUCHY'S THEOREM,

$$\int_{C_1, C_2} f(z) dz = \int_{C_3} f(z) dz = \pi$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi}$$

CHECK:

$$\text{CRC HAS } \int_0^{\infty} \frac{x^{m+1} dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}, \quad 0 < m < n$$

TRY $m=1, n=2$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2 \sin \frac{\pi}{2}} = \frac{\pi}{2}$$

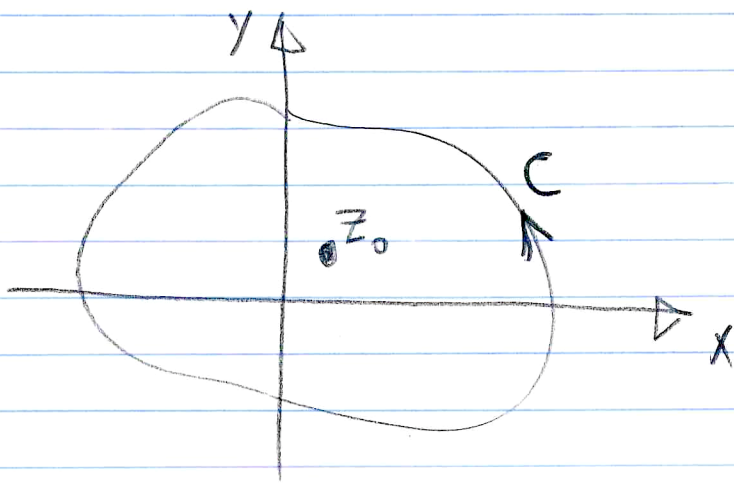
$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

ALTERNATIVELY,

$$\int \frac{dx}{1+x^2} = \text{TAN}^{-1} x$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \text{TAN}^{-1} x \Big|_{-\infty}^{\infty} = \text{TAN}^{-1} \infty - \text{TAN}^{-1}(-\infty) \\ &= \frac{\pi}{2} - -\frac{\pi}{2} = \pi \end{aligned}$$

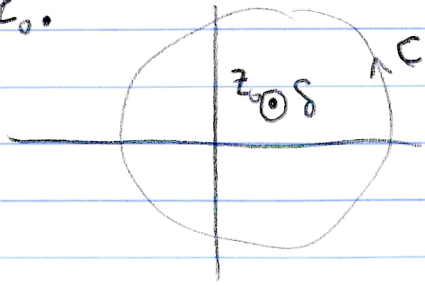
§ 24.10 CAUCHY'S INTEGRAL FORMULA



IF z_0 IS INSIDE C AND $f(z)$ IS ANALYTIC WITHIN AND ON C , THEN

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

PROOF: CONSTRUCT AN INFINITESIMAL LOOP, S , ENCLOSING z_0 .



ACCORDING TO THE RESULT ON THE LAST PAGE,

$$\oint_C \frac{f(z)}{z-z_0} dz = \oint_S \frac{f(z)}{z-z_0} dz = f(z_0) \oint_S \frac{1}{z-z_0} dz$$

IF S IS A CIRCLE CENTERED AT z_0 , THEN OUR VERY FIRST EXAMPLE GIVES US THE RESULT:

$$\oint_S \frac{1}{z-z_0} dz = 2\pi i. \quad \underline{\text{QED.}}$$

IF WE DIFFERENTIATE CAUCHY'S INTEGRAL FORMULA W.R.T. z_0 :

$$\frac{dF(z_0)}{dz_0} = \frac{1}{2\pi i} \frac{\partial}{\partial z_0} \oint_C \frac{f(z)}{z-z_0} dz$$

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

AND IT ALSO FOLLOWS THAT

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

IMPLICATION:

SUPPOSE THAT YOU KNOW $f(z)$ ONLY ON C . THEN CAUCHY'S FORMULA TELLS YOU $f(z)$ FOR ANY INTERIOR POINT.

Q: IS THIS SURPRISING?

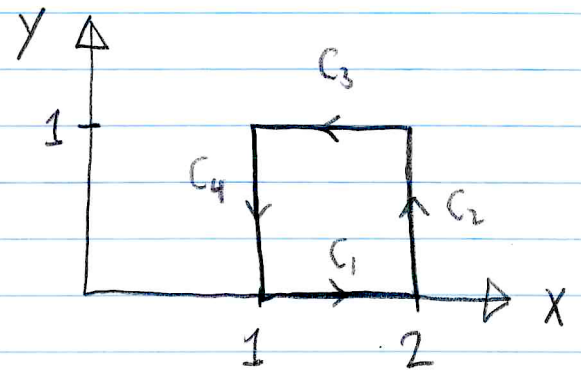
A: CONSIDER $\nabla^2 \Phi = 0$. GIVEN $\Phi(x, y)$ ON THE BOUNDARY, IS THAT ENOUGH TO GET Φ ON THE INTERIOR? WHY IS THIS ANALOGOUS?

EXAMPLE

$$f(z) = \frac{1}{z}$$

VERIFY EXPLICITLY THAT

$$\oint f(z) dz = 0.$$



FOR THE CONTOUR SHOWN.

$$z = x + iy, \quad dz = dx + i dy$$

$$I_1 = \int_{C_1} \frac{dz}{z} = \int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2$$

$$I_2 = \int_{C_2} \frac{dz}{z} = \int_0^1 \frac{i dy}{2 + iy} = \ln(2 + iy) \Big|_0^1 = \ln\left(\frac{2+i}{2}\right)$$

$$I_3 = \int_{C_3} \frac{dz}{z} = \int_2^1 \frac{dx}{x + i} = \ln(x + i) \Big|_2^1 = \ln\left(\frac{1+i}{2+i}\right)$$

$$I_4 = \int_{C_4} \frac{dz}{z} = \int_1^0 \frac{i dy}{1 + iy} = \ln(1 + iy) \Big|_1^0 = \ln\left(\frac{1}{1+i}\right)$$

$$\oint \frac{dz}{z} = \sum I_i = \ln \left[2 \left(\frac{2+i}{2}\right) \left(\frac{1+i}{2+i}\right) \left(\frac{1}{1+i}\right) \right]$$

$$= \ln 1 = 0 \quad \checkmark$$