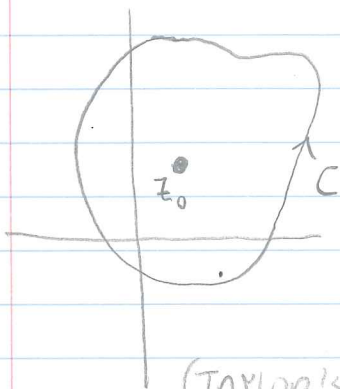


§ 24.11 TAYLOR & LAURENT SERIES

①



IF $f(z)$ ANALYTIC INSIDE & ON C , THEN

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \text{ AND}$$

$$a_n = f^{(n)}(z_0) / n!$$

(TAYLOR'S THEOREM — PROOF IN RHB)

→ SUPPOSE INSTEAD THAT $f(z)$ HAS A POLE OF ORDER p AT $z=z_0$. THEN

$$g(z) = (z-z_0)^p f(z)$$

IS ANALYTIC IN & ON C . THEN

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^n$$

BUT CAUCHY'S INTEGRAL FORMULA (24.48) TELLS US

$$g^{(n)} = \frac{n!}{2\pi i} \oint_C \frac{g(z) dz}{(z-z_0)^{n+1}} = \frac{n!}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n-p+1}}$$

SO

$$g(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n-p+1}}$$

AND SO

$$f(z) = g(z) (z-z_0)^{-p} = \sum_{m=-p}^{\infty} a_m (z-z_0)^m, \quad (m=n-p)$$

$$a_m = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{m+1}} dz$$

NOTE THAT THE a_m ARE UNIQUE!

- THE TERMS $m = -p, -p+1, \dots, -1$ ARE CALLED THE PRINCIPAL PART.

- THE TERMS $m = 0, 1, \dots$ ARE CALLED THE ANALYTIC PART.

- a_{-1} (NOT a_{-p} !) IS CALLED THE RESIDUE OF $f(z)$ AT z_0 :

$$R(z_0) = \frac{1}{2\pi i} \oint_C f(z) dz = a_{-1}$$

CLAIM: ANY CLOSED CURVE CONTAINING z_0 , BUT NOT CONTAINING ANY OTHER SINGULARITIES! (DISCUSS)

FROM THIS FOLLOWS THE RESIDUE THEOREM, (§ 24.12)

IF $f(z)$ IS ANALYTIC INSIDE & ON C EXCEPT FOR A FINITE # OF POLES (AT z_j), THEN

$$\oint_C f(z) dz = 2\pi i \sum_j R(z_j)$$

WRITTEN R_j IN RHB

WE CONCLUDE THAT IT IS WORTH LEARNING TO CALCULATE RESIDUES!

EXAMPLE (RHB P. 856)

$$f(z) = \frac{1}{z(z-2)^3}$$

FIND LAURENT SERIES EXPANSION & RESIDUE ABOUT EACH POLE.

(a) $z=0$: (SIMPLE POLE):

$$f(z) = \frac{1}{z} (z-2)^{-3}$$

↳ CAN EXPAND ABOUT $z=0$ USING BINOMIAL THEOREM.

$$f(z) = \frac{1}{z} \left(-\frac{1}{2}\right) \left(1 - \frac{z}{2}\right)^{-3} = \frac{-1}{8z} \left[1 + (-3) \left(-\frac{z}{2}\right) + \frac{(-3)(-4)}{2!} \left(-\frac{z}{2}\right)^2 + \frac{(-3)(-4)(-5)}{3!} \left(-\frac{z}{2}\right)^3 + \dots \right]$$

$$= \frac{-1}{8z} - \frac{3}{16} - \frac{3z}{16} - \frac{5z^2}{32} - \dots$$

RESIDUE $R(0) = -\frac{1}{8}$

(b) $z=2$: (POLE OF ORDER 3):

LET $\xi = z-2$,

$$f(z) = f(\xi+2) = \frac{1}{(\xi+2)\xi^3} = \frac{1}{2\xi^3} \left(1 + \frac{\xi}{2}\right)^{-1}$$

$$= \frac{1}{2\xi^3} \left[1 - \frac{\xi}{2} + \frac{(-1)(-2)}{2!} \left(\frac{\xi}{2}\right)^2 + \frac{(-1)(-2)(3)}{3!} \left(\frac{\xi}{2}\right)^3 + \dots \right]$$

$$= \frac{1}{2\xi^3} \left[1 - \frac{\xi}{2} + \left(\frac{\xi}{2}\right)^2 - \left(\frac{\xi}{2}\right)^3 + \left(\frac{\xi}{2}\right)^4 - \dots \right]$$

$$= \frac{1}{2\xi^3} - \frac{1}{4\xi^2} + \frac{1}{8\xi} - \dots$$

↳ $R(2) = \frac{1}{8}$

A HANDY TRICK FOR EVALUATING RESIDUES (P. 857)

GIVEN $f(z)$ WITH A POLE OF ORDER p AT z_0 ,

$$f(z) = \frac{a_{-p}}{(z-z_0)^p} + \frac{a_{-p+1}}{(z-z_0)^{p-1}} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

$$(z-z_0)^p f(z) = a_{-p} + a_{-p+1}(z-z_0) + \dots + a_{-1}(z-z_0)^{p-1} + \dots$$

TAKE $\frac{d}{dz}^{p-1}$ OF BOTH SIDES:

$$\left(\frac{d}{dz}\right)^{p-1} [(z-z_0)^p f(z)] = a_{-1} (p-1)! + \sum_{j=1}^{\infty} b_j (z-z_0)^j$$

TAKE LIMIT $z \rightarrow z_0$:

$$a_{-1} = R(z_0) = \lim_{z \rightarrow z_0} \left\{ \frac{1}{(p-1)!} \left(\frac{d}{dz}\right)^{p-1} [(z-z_0)^p f(z)] \right\} \quad (24.56)$$

FOR A SIMPLE POLE THEN,

$$R(z_0) = \lim_{z \rightarrow z_0} [(z-z_0) f(z)] \quad (24.57)$$

EXAMPLES

① TEST (24.56) ON THE PREVIOUS EXAMPLE, $f(z) = \frac{1}{z(z-2)^3}$

(a) FOR $z=0$, (SIMPLE POLE)

$$R(0) = \lim_{z \rightarrow 0} \left\{ \frac{z}{z(z-2)^3} \right\} = -\frac{1}{8} \checkmark$$

(b) FOR $z=2$, (POLE OF ORDER 3)

$$\begin{aligned} R(2) &= \lim_{z \rightarrow 2} \left\{ \frac{1}{z} \frac{d^2}{dz^2} \frac{1}{z} \right\} \\ &= \lim_{z \rightarrow 2} \left\{ \frac{1}{z} \frac{d}{dz} \left(\frac{-1}{z^2} \right) \right\} \\ &= \lim_{z \rightarrow 2} \left\{ \frac{1}{z} \frac{2}{z^3} \right\} = \frac{1}{8} \checkmark \end{aligned}$$

② FOR $f(z) = \frac{e^{iz}}{(z^2+1)^2}$, FIND $R(i)$

$$f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2}$$

$$\begin{aligned} R(i) &= \lim_{z \rightarrow i} \left\{ \frac{d}{dz} \frac{e^{iz}}{(z+i)^2(z-i)^2} \right\} \\ &= \frac{i e^{iz}}{(z+i)^2} - 2 \frac{e^{iz}}{(z+i)^3} \\ &= \frac{i e^{iz}}{(2i)^2} - \frac{2 e^{iz}}{(2i)^3} \\ &= \frac{i}{-4} - \frac{2}{-8ie} \\ &= \frac{-i}{4e} + \frac{i}{4e} \end{aligned}$$

$$R(i) = -\frac{i}{2e}$$

EXAMPLE

$$\oint_C \frac{z dz}{(z-1)(z-2i)}$$

(a) where C is $|z-1|=1$

(b) C is $|z-i|=2$

(c) C is $|z|=10$

A-Dog asked if the residue theorem can be understood in terms of Stokes' theorem, $\oint_{\partial S} \vec{B} \cdot d\vec{r} = \int_S \nabla \times \vec{B} \cdot d\vec{A}$.

Answer: Both Stokes' theorem and Gauss's theorem are involved! Consider an analytic function $f(z) = u(x,y) + iv(x,y)$.

Recall that a vector field satisfying $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{B} = 0$ could be constructed by using u or v as scalar potential (and then the other is a stream function). There is also a completely different vector field that can be built from f :

$$\text{Let } \vec{B} = v\hat{x} + u\hat{y}; \quad f = B_y + iB_x$$

Quickly rederive the Cauchy-Euler relations:

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial(iy)} + i \frac{\partial v}{\partial(iy)}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}; & \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= 0 & \Rightarrow & \nabla \times \vec{B} = 0 \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} &= 0 & \Rightarrow & \nabla \cdot \vec{B} = 0 \end{aligned}$$

In this context, what is the interpretation of the closed contour integral?

$$\oint_C f(z) dz = \oint_C (B_y + i B_x) (dx + i dy)$$

$$= \oint_C \left[(B_y dx - B_x dy) + i (B_x dx + B_y dy) \right]$$

$$\vec{dl} = \hat{x} dx + \hat{y} dy$$

$\hat{n} = (\hat{x} dy - \hat{y} dx) / dl$, normal to the curve

$$= \oint_C (\vec{B} \cdot \hat{n}) dl + i \oint_C \vec{B} \cdot \vec{dl};$$

Using Gauss's Theorem on the first integral and Stokes' Theorem on the second,

$$\oint_C f(z) dz = \int_A (\nabla \cdot \vec{B}) dA + i \int_A (\nabla \times \vec{B}) \cdot d\vec{A} = 0,$$

where A is the area enclosed by C (i.e. $C = \partial A$),

provided f analytic inside & on C (Cauchy's theorem!).

Invoking the residue theorem,

$$\begin{aligned} \oint_C f(z) dz &= 2\pi i \sum_{z_j \in A} R(z_j) \\ &= \int_A (\nabla \cdot \vec{B}) dA + i \int_A (\nabla \times \vec{B}) \cdot d\vec{A} \end{aligned}$$

Exercise: repeat the above using

$$\vec{B} = u\hat{x} - v\hat{y}; \quad f = B_x - iB_y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \Rightarrow \nabla \cdot \vec{B} = 0$$
$$-\frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} = 0 \Rightarrow \nabla \times \vec{B} = 0$$

$$\oint_c f(z) dz = \oint_c (B_x - iB_y)(dx + i dy)$$
$$= \oint_c (B_x dx + B_y dy) + i \oint_c (B_x dy - B_y dx)$$
$$= \oint_c \vec{B} \cdot d\vec{l} + i \oint_c (\vec{B} \cdot \hat{n}) dl$$
$$= \int_A (\nabla \times \vec{B}) \cdot d\vec{A} + i \int_A (\nabla \cdot \vec{B}) dA$$

↑
Role reversal!
↑

So, the result for the real or imaginary part of the integral can be obtained using either Stokes' or Gauss's theorem.

Q: Are Stokes' theorem and Gauss's theorem somehow equivalent? Discuss!