

§ 24.13 — APPLYING CONTOUR INTEGRATION TO DEFINITE INTEGRALS

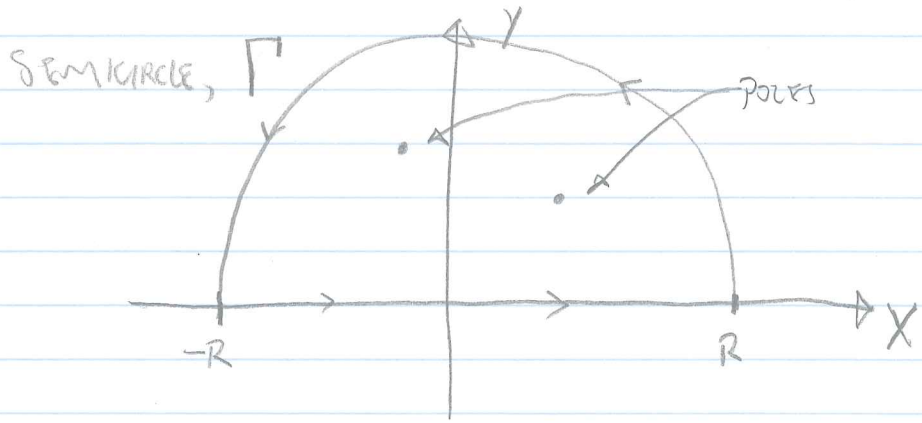
ONE OF THE MOST IMPORTANT APPLICATIONS IS TO IMPROPER INTEGRALS, E.G.

$$\int_{-\infty}^{\infty} f(x) dx$$

WE WRITE THIS AS

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx,$$

STRATEGY: CONVERT TO A CONTOUR INTEGRAL



$$\int_{-\infty}^{\infty} f(x) dx = \oint_C f(z) dz - \int_{\Gamma} f(z) dz$$

\uparrow
 SEMICIRCLE
 PLUS $-R < x < R$
 ON X-AXIS

$$= \sum_{\text{INSIDE}} R - \int_{\Gamma} f(z) dz$$

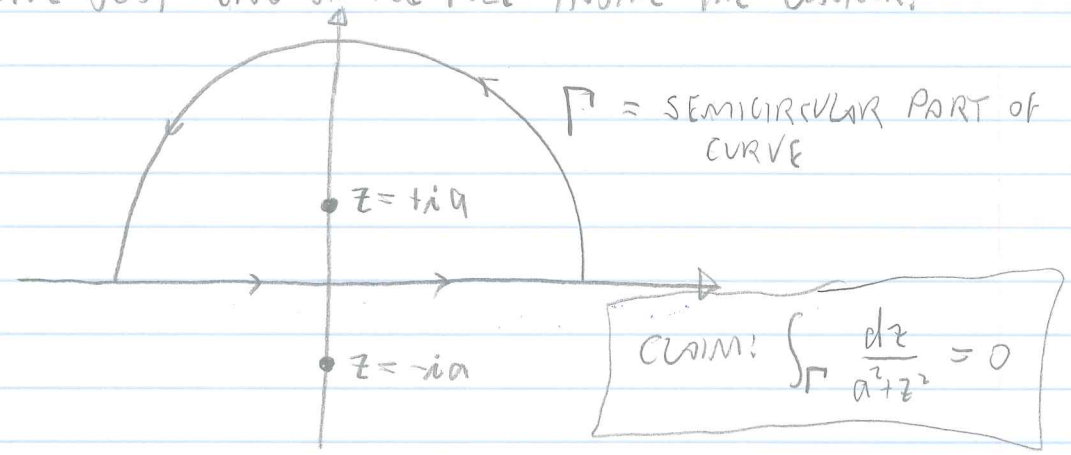
↑ HOPE THIS IS EASIER THAN THE ORIGINAL INTEGRAL!

EXAMPLE EVALUATE:

$$I = \int_{-\infty}^{\infty} \frac{dx}{a^2+x^2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \oint_C \frac{dz}{(z+ia)(z-ia)} - \int_{\Gamma} \frac{dz}{a^2+z^2}$$

WE HAVE JUST ONE SIMPLE POLE INSIDE THE CONTOUR:



$$R(ia) = \lim_{z \rightarrow ia} \left\{ \frac{1}{z+ia} \right\} = \frac{1}{2ia}$$

$$I = 2\pi i \sum_{\text{INSIDE}} R = \frac{2\pi i}{2ia}$$

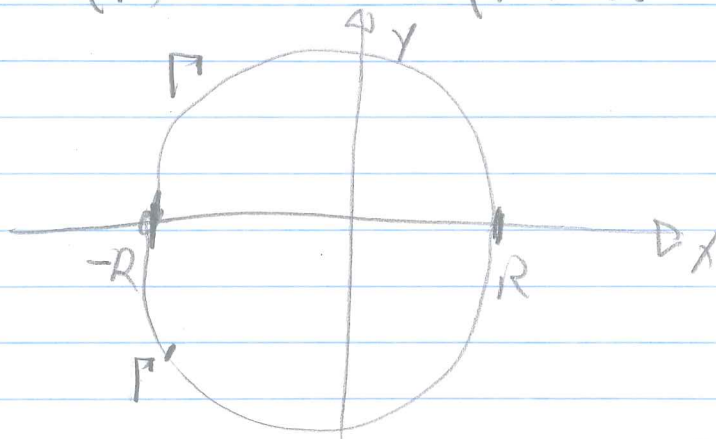
$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{a}$$

CONSIDER THE F.T. INTEGRAL

$$\begin{aligned} \mathcal{F}\left(\frac{1}{1+x^2}\right) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_L \frac{e^{-ikz}}{1+z^2} dz \end{aligned}$$

WHERE CONTOUR L IS THE REAL AXIS.

FOR $k > 0$, THE CONTOUR CAN BE CLOSED WITH A SEMICIRCLE, ALONG WHICH THE CONTOUR INTEGRAL IS ZERO. SHOULD I CHOOSE THE UPPER (Γ) OR LOWER (Γ') SEMICIRCLE?



EVALUATE \oint , AND HENCE FIND $\mathcal{F}\left(\frac{1}{1+x^2}\right)$ FOR $k > 0$.

FIND $\mathcal{F}\left(\frac{1}{1+x^2}\right)$ FOR $k < 0$

COMBINE.

Clearer explanation:

OUR NUMERATOR, e^{ikx} , LOOKS INNOCENT.
JUST A COMPLEX NUMBER OF UNIT MODULUS, RIGHT?

Q: WHAT'S DIFFERENT ABOUT e^{ikz} ?

A: LOOK AT $z = r \cos \theta + i r \sin \theta$. THEN

$$e^{-ikhz} = e^{-ikr \cos \theta} e^{kr \sin \theta}$$

↑
BLOWS UP AS $r \rightarrow \infty$,
FOR $0 < \theta < \pi$, UNLESS $k < 0$.

SO INTEGRATE ON Γ' FOR $k < 0$

NOW, FOR $k > 0$, INTEGRATE ON Γ' SO $\sin \theta \leq 0$.

THE RESULT SHOULD BE SOMETHING LIKE

$$\mathcal{F}_1 \left[\frac{1}{x^2+1} \right] = \begin{cases} \sqrt{\frac{\pi}{2}} e^{2k}, & k < 0 \\ \sqrt{\frac{\pi}{2}} e^{-2k}, & k > 0 \end{cases}$$

$$= \sqrt{\frac{\pi}{2}} e^{-2|k|}$$

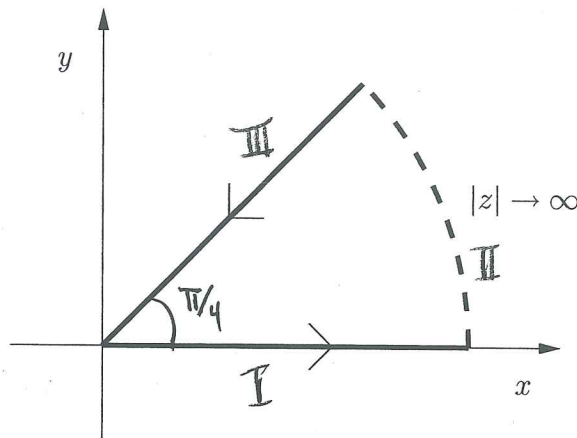
EXAMPLE EVALUATE

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 4)}$$

$$\text{Roots: } z = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm i\sqrt{3}$$

(THESE ARE SIMPLE POLES)

EXAMPLE



6. Let $f(z) \equiv 1/(1+z^2)$.

(a) (5 points) Where is $f(z)$ analytic?

(b) (15 points) It is easy to show using trigonometric substitution that

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Use this fact and integrate $f(z)$ on the illustrated contour to show that

$$\int_0^{\infty} \frac{1+r^2}{1+r^4} dr = \frac{\pi}{2\sqrt{2}} \text{ and } \frac{\pi}{\sqrt{2}}$$

$$\int_0^{\infty} \frac{1-r^2}{1+r^4} dr = 0.$$

(a) $f(z)$ IS ANALYTIC EVERYWHERE EXCEPT AT THE ROOTS OF THE DENOMINATOR, $z = \pm i$.

$$(b) 0 = \oint f(z) dz = \int_I f(z) dz + \int_{II} f(z) dz + \int_{III} f(z) dz$$

$$0 = \int_0^{\infty} \frac{dx}{1+x^2} + 0 + \int_{III} f(z) dz \quad \star$$

ON PATH III, $dz = e^{i\pi/4} dr$; $z^2 = x^2 - y^2 + 2ixy = 2ix^2$
 $r^2 = x^2 + y^2 = 2x^2$

$$\text{SO } \int_{III} f dz = - \int_{r=0}^{\infty} \frac{e^{i\pi/4} dr}{1+ir^2}$$

CONT.

scratch paper

$$\begin{aligned}\int_{\text{III}} f dz &= - \int_0^{\infty} \frac{\frac{1}{\sqrt{z}}(1+i)}{1+ir^2} dr \\ &= -\frac{1}{\sqrt{2}} \int_0^{\infty} \frac{(1+i)(1-ir^2)}{1+r^4} dr \\ &= -\frac{1}{\sqrt{2}} \int_0^{\infty} \frac{[1+r^2+i(1-r^2)]}{1+r^4} dr\end{aligned}$$

PLUGGING BACK INTO \star , THE REAL PART IS:

$$\int_0^{\infty} \frac{1+r^2}{1+r^4} dr = \frac{\pi}{\sqrt{2}} \quad \checkmark$$

AND IMAGINARY PART:

$$\int_0^{\infty} \frac{1-r^2}{1+r^4} dr = 0 \quad \checkmark$$

JORDAN'S LEMMA

NOTE THAT $\int_{\Gamma} \rightarrow 0$ WAS VERY IMPORTANT.

WE NOW EXPAND THE SET OF SUCH SEMICIRCLES THAT INTEGRATE TO ZERO.

IF ① $f(z)$ IS ANALYTIC FOR $y > 0$ EXCEPT FOR A FINITE # POLES AND ② $\max |f(z)| \rightarrow 0$ AS $|z| \rightarrow \infty$ FOR $y > 0$ AND ③ $m > 0$,

$$I_{\Gamma} = \int_{\Gamma} e^{imz} f(z) dz \rightarrow 0 \text{ AS } R \rightarrow \infty$$

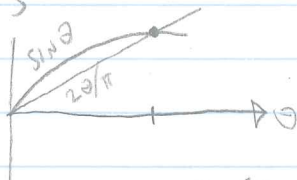
PROOF

$$I_{\Gamma} \leq \int_{\Gamma} \overbrace{|e^{imz}|}^{R \sin \theta} \overbrace{|f(z)|}^M |dz| \leq \overbrace{\max(f)}^M \int_0^{\pi} |e^{imz}| R d\theta$$

$$\text{NOW, } |e^{imz}| = |e^{imx} e^{-my}| = e^{-my} = e^{-mR \sin \theta}$$

$$\Rightarrow I_{\Gamma} \leq MR \int_0^{\pi} e^{-mR \sin \theta} d\theta = 2MR \int_0^{\pi/2} e^{-mR \sin \theta} d\theta$$

$$\text{ON } 0 < \theta < \pi/2, \sin \theta \geq \frac{2\theta}{\pi},$$



$$\text{SO } e^{-mR \sin \theta} \leq e^{-mR (2\theta/\pi)}$$

$$\therefore I_{\Gamma} \leq 2MR \int_0^{\pi/2} e^{-2mR \theta/\pi} d\theta = \frac{\pi M}{m} (1 - e^{-mR}) < \frac{\pi M}{m}$$

BUT RECALL $M \rightarrow 0$, SO

$$\lim_{R \rightarrow \infty} I_{\Gamma} = 0 \quad \underline{\text{QED}}$$