

OUR TOOLS FOR CONTOUR INTEGRATION ARE:

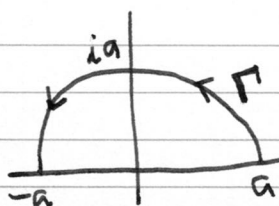
① $\oint_C f(z) dz = 2\pi i \sum_j R(z_j)$ (RESIDUE THEOREM)

② $R(z_0) = \lim_{z \rightarrow z_0} \left\{ \frac{1}{(p-1)!} \left(\frac{d}{dz} \right)^{p-1} \left[(z-z_0)^p f(z) \right] \right\}$
 SIMPLE POLE ($p=1$): $R(z_0) = \lim_{z \rightarrow z_0} \left[(z-z_0) f(z) \right]$

③ IF $f(z)$ ANALYTIC FOR $y > 0$ EXCEPT FOR A FINITE # OF POLES
 & $\max |f(z)| \rightarrow 0$ AS $|z| \rightarrow \infty$ FOR $y > 0$
 & $m > 0$
 THEN

$$\int_{\Gamma} e^{imz} f(z) dz \rightarrow 0$$

FOR THE CONTOUR Γ IN THE LIMIT $a \rightarrow \infty$:



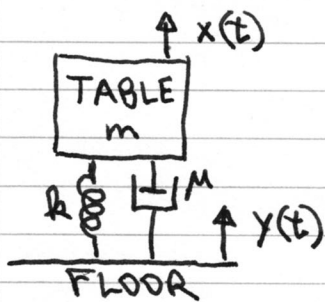
(JORDAN'S LEMMA)

MANY OF OUR EXAMPLES ARE INTEGRALS THAT COME DOWN TO A NUMBER, E.G.

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

NEXT WE DO AN EXAMPLE THAT PRODUCES SOME PHYSICAL INSIGHT VIA AN INTEGRAL THAT IS NOT TRIVIAALLY REDUCIBLE TO A DIMENSIONLESS MATHEMATICAL OBJECT.

HOW DOES A FLOATING OPTICAL TABLE RESPOND TO FLOOR VIBRATION? ①



EQUATION OF MOTION:

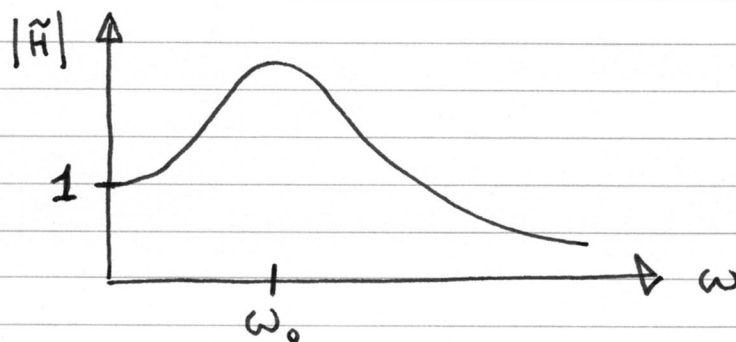
$$m\ddot{x} = k(y-x) + M(\dot{y}-\dot{x})$$

$$\text{LET } \omega_0 \equiv \sqrt{\frac{k}{m}}, \quad \omega_1 \equiv \frac{M}{2m}$$

THEN FOURIER TRANSFORM:

$$\tilde{x}(\omega) = \left(\frac{\omega_0^2 + 2i\omega_1\omega}{-\omega^2 + 2i\omega_1\omega + \omega_0^2} \right) \tilde{y}(\omega)$$

↑
FREQUENCY RESPONSE $\tilde{H}(\omega)$



IF $y(t) = \delta(t)$, THEN $\tilde{y}(\omega) = \frac{1}{\sqrt{2\pi}}$, AND $x(t) \rightarrow \frac{H(t)}{\sqrt{2\pi}}$.

IN THIS CASE, I WOULD MAKE A DISTINCTION BETWEEN $H(t)/\sqrt{2\pi}$ (THE IMPULSE RESPONSE) AND THE GREEN'S FUNCTION. BUT IT SHOULD BE APPARENT THAT

$$x(t) = \frac{1}{\sqrt{2\pi}} H(t) * y(t).$$

(DISCUSS)

Now,

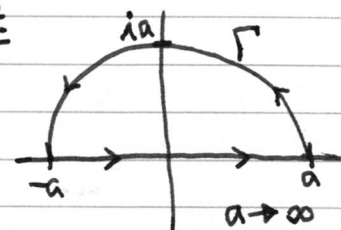
$$\sqrt{2\pi} H(t) = \int_{-\infty}^{\infty} \frac{2i\omega_1\omega + \omega_0^2}{-\omega^2 + 2i\omega_1\omega + \omega_0^2} e^{i\omega t} d\omega.$$

FOR $t < 0$, WE ANTICIPATE BY CAUSALITY THAT $H = 0$.

FOR $t > 0$,

$$2\pi i \sum R(z_j) = \sqrt{2\pi} H(t) + \int_{\Gamma} \frac{(2i\omega_1 z + \omega_0^2) e^{izt}}{-z^2 + 2i\omega_1 z + \omega_0^2} dz$$

BY JORDAN'S LEMMA, $\int_{\Gamma} \rightarrow 0$, AND



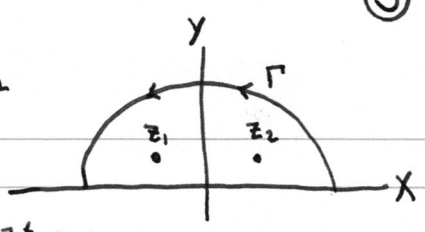
$$\therefore \frac{H(t)}{\sqrt{2\pi}} = i \sum_j R(z_j)$$

THE POLES z_j ARE AT $z = \frac{-2i\omega_1 \pm \sqrt{4\omega_0^2 - 4\omega_1^2}}{-2} = i\omega_1 \pm \sqrt{\omega_0^2 - \omega_1^2}$

- THERE ARE 3 CASES:
- ① UNDERDAMPED, $\omega_1 < \omega_0$, 2 POLES IN $z > 0$
 - ② OVERDAMPED, $\omega_1 > \omega_0$, 2 POLES IN $z > 0$
 - ③ CRITICALLY DAMPED, $\omega_1 = \omega_0$,
ONE SECOND ORDER POLE IN $z > 0$

CASE ① UNDERDAMPED, $\omega_1 < \omega_0$, POLES AT $z = i\omega_1 \pm \omega_2$

WHERE $\omega_2 \equiv \sqrt{\omega_0^2 - \omega_1^2} \in \mathbb{R}^+$.



$$R(z_1) = \lim_{z \rightarrow i\omega_1 - \omega_2} \left[(z - i\omega_1 + \omega_2) \frac{(2i\omega_1 z + \omega_0^2) e^{izt}}{(-z^2 + 2i\omega_1 z + \omega_0^2)} \right]$$

$$= \frac{2i\omega_1 (i\omega_1 - \omega_2) + \omega_0^2}{(i\omega_1 - \omega_2) - i\omega_1 - \omega_2} e^{i(i\omega_1 - \omega_2)t}$$

$$= \left(-\frac{\omega_1^2}{\omega_2} - i\omega_1 + \frac{\omega_0^2}{2\omega_2} \right) e^{-\omega_1 t} e^{-i\omega_2 t}$$

SIMILARLY,

$$R(z_2) = \left(+\frac{\omega_1^2}{\omega_2} - i\omega_1 - \frac{\omega_0^2}{2\omega_2} \right) e^{-\omega_1 t} e^{+i\omega_2 t}$$

↪ CAN SEE THIS BY SUBBING $\omega_2 \rightarrow -\omega_2$ IN $R(z_1)$.

NOW, THE IMPULSE RESPONSE IS:

$$\frac{H(t)}{\sqrt{2\pi}} = i \sum_j R(z_j)$$

$$= i \left[-i \frac{\omega_0^2 - 2\omega_1^2}{\omega_2} e^{-\omega_1 t} \sin(\omega_2 t) - 2i\omega_1 e^{-\omega_1 t} \cos(\omega_2 t) \right]$$

$$= \frac{\omega_0^2 - 2\omega_1^2}{\omega_2} e^{-\omega_1 t} \sin(\omega_2 t) + 2\omega_1 e^{-\omega_1 t} \cos(\omega_2 t)$$

MAINLY DUE TO FORCE TRANSMITTED TO TABLE THROUGH SPRING
DUE TO FORCE TRANSMITTED TO TABLE THROUGH DASHPOT

- FREE OSCILLATIONS ARE AT FREQUENCY $\omega_2 < \omega_0$.
- CONSIDER THE LIMIT $\omega_1 \rightarrow 0$.
- NOTE THAT THE UNITS (FREQUENCY) MATCH OUR $y(t) = \delta(t)$ IMPULSE.

DISCUSS OVERDAMPED AND CRITICALLY DAMPED CASES:

- HOW MANY POLES AND WHERE ARE THEY?
- ARE THEY ALWAYS SIMPLE POLES?

GIVEN AN INPUT POWER SPECTRUM, $\tilde{y}^* \tilde{y}$, COULD YOU FIND THE RMS ACCELERATION OF THE TABLE?