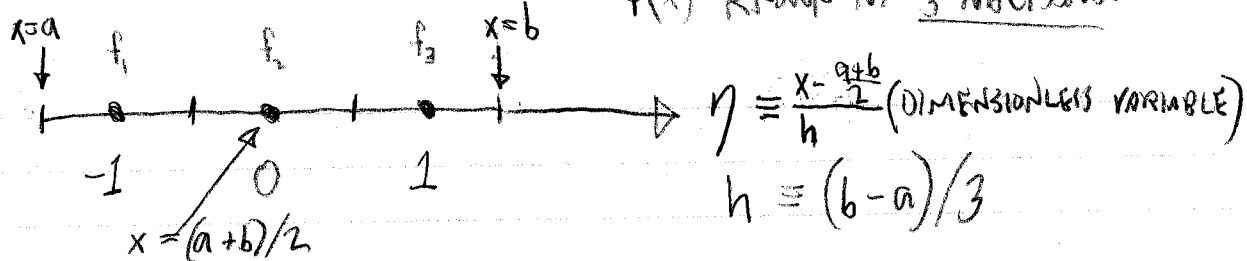


$$\int_a^b f(x) dx = ?$$

①

A HIGHER-ORDER METHOD.

$f(x)$ KNOWN AT 3 ABSCISSAS.



LET'S FIND A PARABOLA THAT WILL FIT OUR 3 POINTS. WE'LL EXPRESS OUR ANSWER IN TERMS OF 3 LAGRANGE INTERPOLATING POLYNOMIALS (NR § 3.1):

① **f_1 TERM** WE WANT A QUADRATIC THAT HAS ROOTS AT $\eta = 0, 1$ AND IS UNITY AT $\eta = -1$.

ROOTS $\rightarrow (\eta - 0)(\eta - 1) = \frac{1}{2}(\eta^2 - \eta)$
 NORMALIZE $\rightarrow \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)}$

② **f_2 TERM** ROOTS AT $\eta = -1, 1$, UNITY AT $x = 0$.

$$\frac{(\eta + 1)(\eta - 1)}{(0 + 1)(0 - 1)} = 1 - \eta^2$$

③ **f_3 TERM** ROOTS AT $\eta = -1, 0$, UNITY AT $\eta = 1$.

$$\frac{(\eta + 1)(\eta - 0)}{(1 + 1)(1 - 0)} = \frac{1}{2}(\eta^2 + \eta)$$

SO THE DESIRED PARABOLA IS

$$y(\eta) = \frac{1}{2}(\eta^2 - \eta)f_1 + (1 - \eta^2)f_2 + \frac{1}{2}(\eta^2 + \eta)f_3$$

SUPPOSING NOW THAT $y(\eta) \approx f(x), \dots$

WE APPROXIMATE THE INTEGRAL:

$$\int_a^b f(x) dx \approx h \int_{-3/2}^{3/2} y(\eta) d\eta$$

EVEN OVER
EVEN LIMITS

$$= h \left[\left(\frac{1}{6} \eta^3 - \frac{1}{4} \eta^2 \right) f_1 + \left(\eta - \frac{1}{3} \eta^3 \right) f_2 + \left(\frac{1}{6} \eta^3 + \frac{1}{4} \eta^2 \right) f_3 \right]_{-3/2}^{3/2}$$

$$= h \left[\frac{1}{6} \left(\frac{27}{8} + \frac{27}{8} \right) f_1 + \left(\frac{6}{2} - \frac{18}{8} \right) f_2 + \frac{1}{6} \left(\frac{27}{8} + \frac{27}{8} \right) \right]$$

$$= h \left[\frac{9}{8} f_1 + \frac{6}{8} f_2 + \frac{9}{8} f_3 \right]$$

THE COEFFICIENTS TOTAL $3h$.
 (WHY DOES THIS MAKE SENSE?)

SO

$$\int_a^b f(x) dx \approx 3h \left[\frac{3}{8} f_1 + \frac{1}{4} f_2 + \frac{3}{8} f_3 \right]$$

I DON'T KNOW IF THIS HAS A NAME. WE'LL CALL IT THE TRES HERMANOS RULE*

QUESTION: WHAT IS THE TRUNCATION ERROR?

ANSWER: $O(f^{(4)}(x) h^5)$.

*IN HONOR OF A TACO STAND THAT I LIKED IN MT.VIEW, CA.

TO FORM AN EXTENDED TRES HERMANOS RULE, WE NEED TO INTEGRATE THE 3 SUB-INTERVALS SEPARATELY:

$$\int_1 f dx \approx h \left[\left(\frac{1}{6}x^3 - \frac{1}{4}x^2 \right) f_1 + \left(x - \frac{1}{3}x^3 \right) f_2 + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 \right) f_3 \right]_{-1/2}^{1/2}$$

$$= h \left\{ \left[\frac{1}{6} \left(-\frac{1}{8} + \frac{27}{8} \right) - \frac{1}{4} \left(\frac{1}{4} - \frac{9}{4} \right) \right] f_1 \right.$$

$$+ \left[\left(-\frac{1}{2} + \frac{3}{2} \right) - \frac{1}{3} \left(-\frac{1}{8} + \frac{27}{8} \right) \right] f_2$$

$$\left. + \left[\frac{1}{6} \left(-\frac{1}{8} + \frac{27}{8} \right) + \frac{1}{4} \left(\frac{1}{4} - \frac{9}{4} \right) \right] f_3 \right\}$$

$$\int_1 f dx \approx \frac{25}{24} h f_1 - \frac{1}{12} h f_2 + \frac{1}{24} h f_3$$

NOTE — COEFFICIENTS SUM TO h.

BY SYMMETRY,

$$\int_3 f dx \approx \frac{1}{24} h f_1 - \frac{1}{12} h f_2 + \frac{25}{24} h f_3$$

FINALLY,

$$\int_2 f dx \approx h \left[\left(\frac{1}{6}x^3 - \frac{1}{4}x^2 \right) f_1 + \left(x - \frac{1}{3}x^3 \right) f_2 + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 \right) f_3 \right]_{-1/2}^{1/2}$$

$$= h \left\{ \frac{1}{6} \left(\frac{1}{4} \right) f_1 + \left[1 - \frac{1}{3} \left(\frac{1}{4} \right) \right] f_2 + \frac{1}{6} \left(\frac{1}{4} \right) f_3 \right\}$$

$$\int_2 f dx \approx \frac{1}{24} h f_1 + \frac{11}{12} h f_2 + \frac{1}{24} h f_3$$

AGAIN — COEFFICIENTS SUM TO h.

Building an Extended *Tres Hermanos* Rule

$$\int_1 f(x) dx \quad \left[\begin{array}{ccc} \frac{25}{24} & \frac{-1}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_2 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{11}{12} & \frac{1}{24} \end{array} \right]$$

$$\int_3 f(x) dx \quad \left[\begin{array}{ccc} \frac{1}{24} & \frac{-1}{12} & \frac{25}{24} \end{array} \right]$$

coefficients: $\left[\begin{array}{ccccccc} \frac{26}{24} & \frac{21}{24} & \frac{25}{24} & 1 & 1 & \frac{25}{24} & \frac{21}{24} & \frac{26}{24} \end{array} \right]$

So THE EXTENDED TR RULE IS:

$$\int_a^b f(x) dx \approx h \left[\frac{26}{24} f_1 + \frac{21}{24} f_2 + \frac{25}{24} f_3 + \sum_{n=4}^{N-3} f_n + \frac{25}{24} f_{N-2} + \frac{21}{24} f_{N-1} + \frac{26}{24} f_N \right], \quad \text{WHERE}$$

$$h = (b-a)/N, \quad f_n = f(a + [n - \frac{1}{2}]h).$$

A POINT OF CURIOSITY:

THIS LOOKS JUST LIKE THE EXTENDED MIDPOINT METHOD, EXCEPT FOR SLIGHTLY DIFFERENT WEIGHTS AT THE FIRST 3 AND LAST 3 TERMS. HOW CAN THIS MATTER MUCH FOR LARGE N ?

AND YET, IF OUR ESTIMATE OF TRUNCATION ERROR IS CORRECT, THEN WE EXPECT THE ERROR OF THIS EXTENDED METHOD TO BE

$$\mathcal{O}(Nh^5) = \mathcal{O}\left(\frac{1}{N^4}\right)$$

PREDICTION: SHOULD CONVERGE MUCH FASTER THAN THE MIDPOINT METHOD.

Q: DO YOU BELIEVE IT?

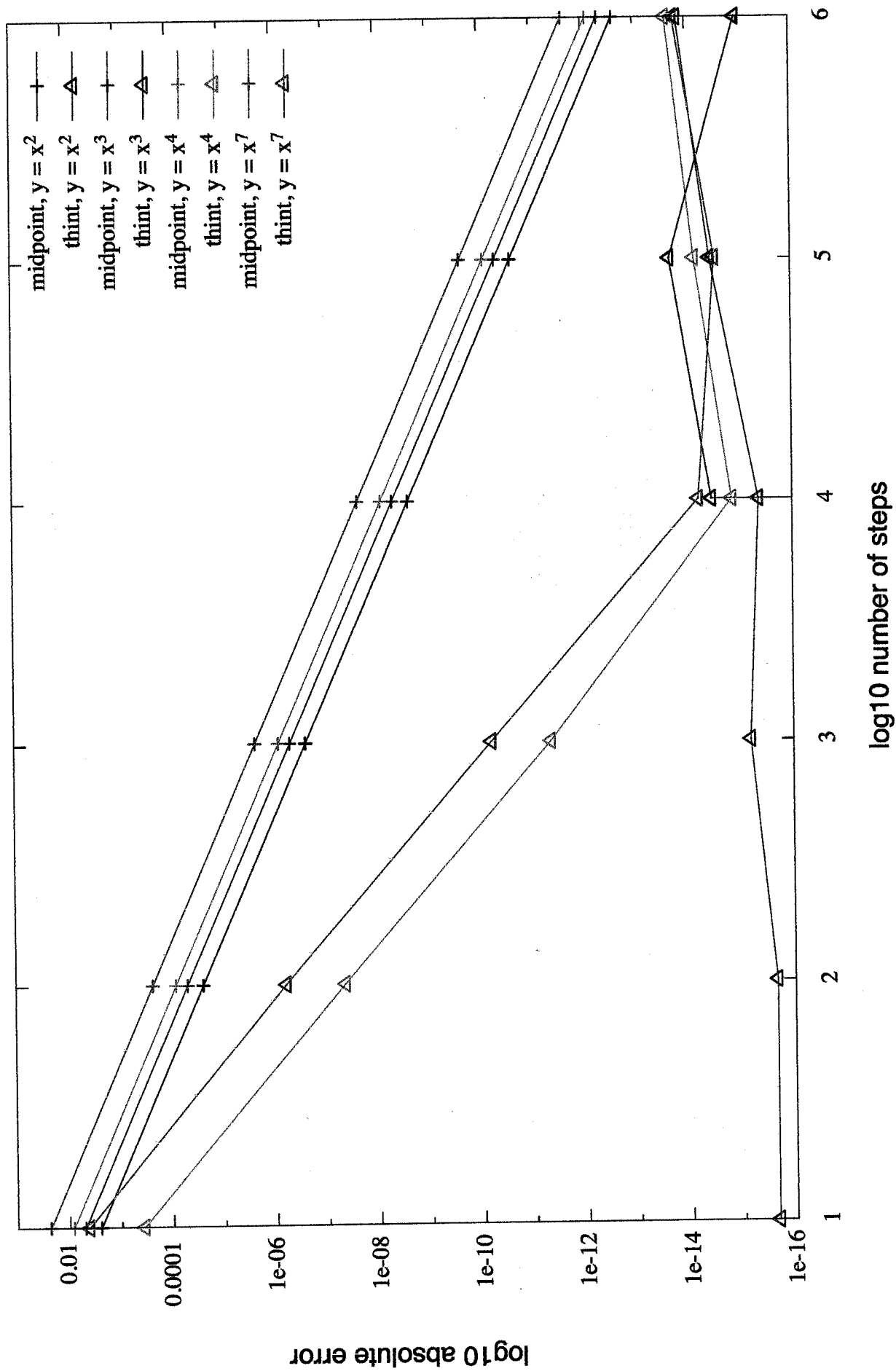
```

% Extended Tres Hermamos Integrator.
% A 3rd order open method, with uniformly spaced abscissas.
% 2009-Feb-01 C. Kankelborg

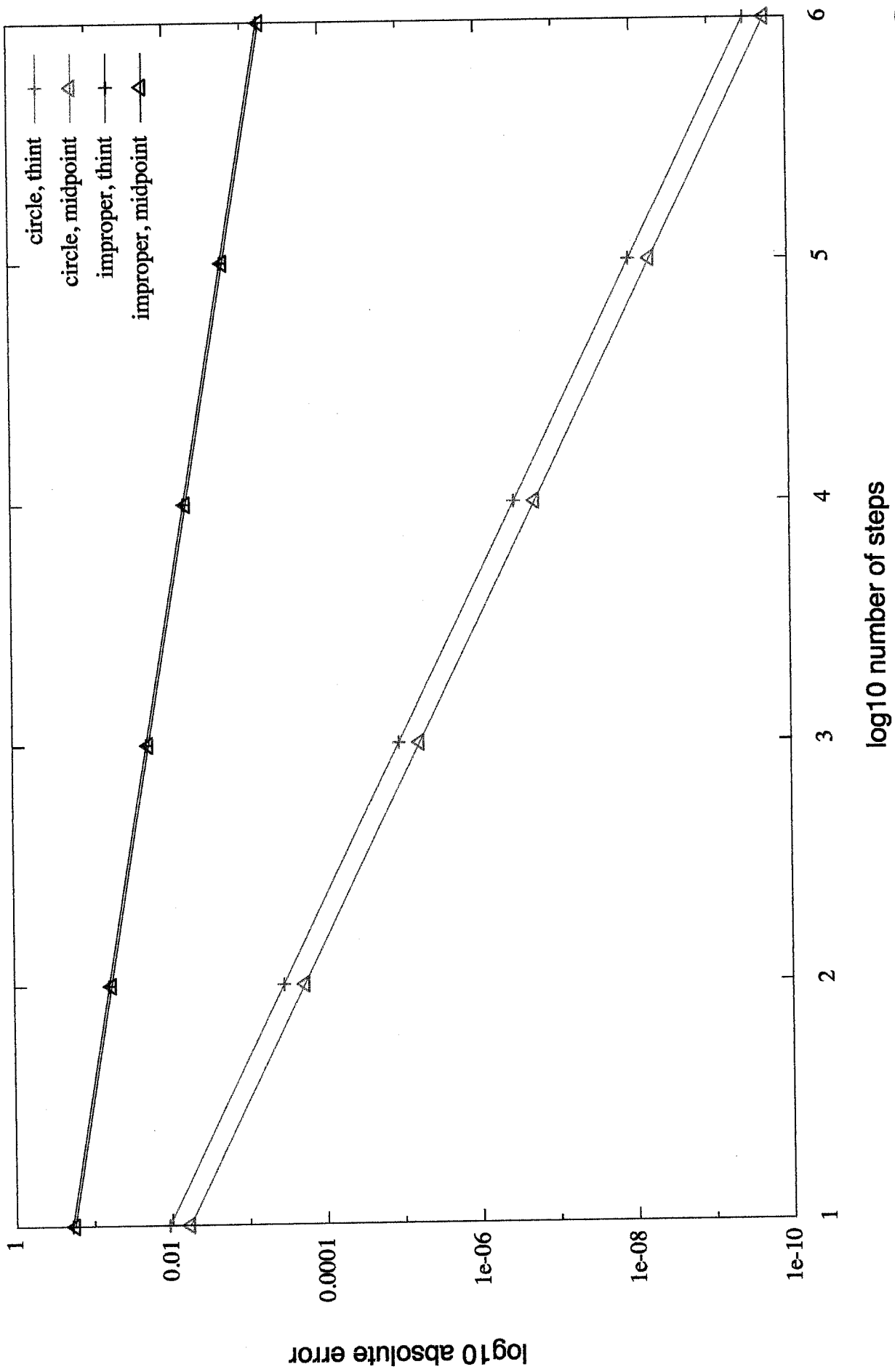
function result = thint(integrand, a, b, N)
    dx = (b-a)/N; % step size for N intervals
    x = a + ((1:N)-0.5)*dx; % abscissas (like midpoint method)
    vals = integrand(x); % integrand must be able to accept an array of x's.
    % Apply coefficient weights.
    vals(1:3) = vals(1:3) .* [26/24, 21/24, 25/24];
    vals(N-2:N) = vals(N-2:N) .* [25/24, 21/24, 26/24];
    result = sum(vals)*dx;

```

Integrating polynomials



Integrating nasty functions



(8)

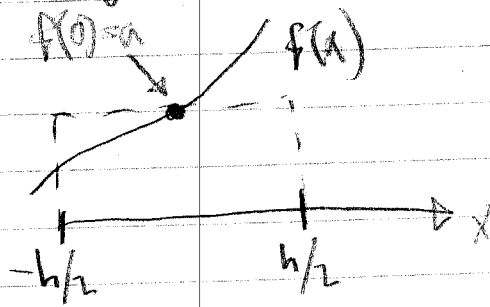
HOW IS IT THAT HIGHER ORDER CORRECTIONS REQUIRE ONLY A FIXED NUMBER (6 FOR TRS HERMANNOS) OUT OF N BE ADJUSTED?

DESPITE ITS FAILURE FOR NON-ANALYTIC ENDPOINTS (PREVIOUS EXAMPLE), TRS HERMANNOS SEEMS UNREASONABLY EFFECTIVE.

WE WILL:

- DERIVE THE TRUNCATION ERROR OF THE MIDPOINT METHOD (FIRST NONZERO TERM)
- DEMONSTRATE THAT THIS TERM CAN BE ESTIMATED FROM DERIVATIVE INFO AT THE ENDPOINTS
- SPECULATE ABOUT ELIMINATING TRUNCATION ERROR ENTIRELY FOR ONE CLASS OF INTEGRALS
- IMPLEMENT & DEMONSTRATE "ERROR-FREE" INTEGRATION USING THE FFT.

TRUNCATION ERROR IN MIDPOINT METHOD



$$f(x) = a + bx + cx^2 + dx^3 + \dots$$

$$\int_{-h/2}^{h/2} f(x) dx = \left[ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3 + \frac{1}{4}dx^4 + \dots \right]_{-h/2}^{h/2}$$

$$= ah + \frac{1}{12}ch^3, \quad c = \frac{f''}{2}$$

MIDPOINT RULE

THIS IS THE LARGEST MISSING PIECE IN THE MIDPOINT RULE.

$$\therefore \text{ERR} = \frac{-1}{24} f'' h^3$$

THIS IS THE TERM WE ARE TRYING TO ELIMINATE WHEN WE DERIVE THE TRIS HERMANOS RULE.

FOR THE EXTENDED MIDPOINT RULE,

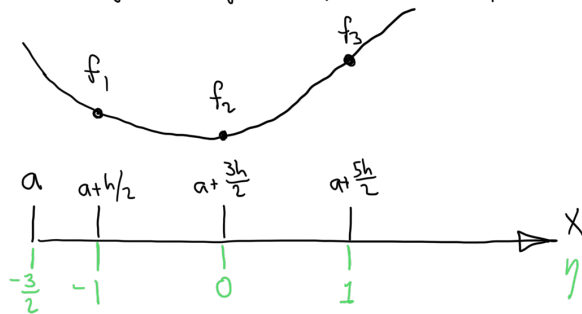
$$\text{ERR}_{\text{EXT}} = \frac{-h^2}{24} \sum \frac{1}{h} f''(x_n) h \approx \frac{-h^2}{24} \int_a^b f''(x) dx$$

$$= \frac{-h^2}{24} [f'(b) - f'(a)]$$

ONLY THE ENDPOINTS MATTER!

Now, we need to estimate $f'(a)$ and $f'(b)$.

Recall our convenient, local coordinate system for fitting 3 adjacent points to a parabola:



$$f(x) \approx \gamma(\eta) = \underbrace{\frac{1}{2}(\eta^2 - \eta)f_1 + (1 - \eta^2)f_2 + \frac{1}{2}(\eta^2 + \eta)f_3}_{\text{interpolating polynomial}}$$

$$f'(x) = \frac{d\eta}{dx} \gamma'(\eta) = \frac{1}{h} \left[(\eta - \frac{1}{2})f_1 - 2\eta f_2 + (\eta + \frac{1}{2})f_3 \right]$$

$$f'(a) = \frac{1}{h} \gamma'(-\frac{3}{2}) = \frac{1}{h} \left[-2f_1 + 3f_2 - f_3 \right]$$

$$\text{By symmetry, } f'(b) = \frac{1}{h} \left[+2f_N - 3f_{N-1} + f_{N-2} \right]$$

The extended midpoint rule truncation error is then

$$ERR_{EXT} = -h \left[\frac{1}{12}(f_1 + f_N) - \frac{3}{24}(f_2 + f_{N-1}) + \frac{1}{24}(f_3 + f_{N-2}) \right]$$

Subtract this error from the midpoint rule:

$$\int_a^b f(x) \approx h \sum_{n=1}^N f_n - ERR_{EXT}$$

$$= h \left[\frac{13}{12} f_1 + \frac{21}{24} f_2 + \frac{25}{24} f_3 + f_4 + f_5 + \dots + f_{N-4} + f_{N-3} + \frac{25}{24} f_{N-2} + \frac{21}{24} f_{N-1} + \frac{13}{12} f_N \right]$$

We have thereby re-derived the extended Trapez Hermans rule!

You will find in textbooks (and on Wikipedia) Simpson's Rule, a closed scheme of the same order as my TH:

$$\int_a^b f(x) dx \approx \frac{h}{3} \left[\overset{f(a)}{f_0} + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{N-2} + 4f_{N-1} + \overset{f(b)}{f_N} \right],$$

Also common is Simpson's $\frac{3}{8}$ Rule:

$$\int_a^b f(x) dx \approx \frac{3h}{8} \left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + \dots + 2f_{N-3} + 3f_{N-2} + 3f_{N-1} + f_N \right]$$

These quadrature rules are more complicated than they need to be.

INTEGRATION OF PERIODIC FUNCTIONS

SUPPOSE WE HAVE:

$$f_n = f(x_n), \quad x_n = a + (n - \frac{1}{2})h,$$

$$h = (b-a)/N$$

$$\left(\text{so } a = x_1 - \frac{h}{2}, \quad b = x_N + \frac{h}{2} \right)$$

NOW, IF $f(x)$ IS AN ANALYTIC, PERIODIC FUNCTION, THEN

$$\begin{aligned} f(b) &= f(a) \\ f'(b) &= f'(a) \\ f''(b) &= f''(a) \\ \text{--- ETC. ---} \end{aligned}$$

WITH \uparrow PERIOD $b-a$

NOW, ANY ERROR TERM OF THE USUAL FORM:

$$\mathcal{O}(f^{(m)} h^{m+1})$$

WILL LEAD TO AN EXTENDED ERROR

$$\text{ERR}_{\text{EXT}} \propto f^{(m-1)}(b) - f^{(m-1)}(a)$$

$$\Rightarrow \text{ERR}_{\text{EXT}} \rightarrow 0 \quad \text{FOR OUR PERIODIC FXN., } \forall m.$$

NOW, WHEN THE LEADING TERM IN THE TRUNCATION ERROR COMES FROM THE m^{th} DERIVATIVE. RECALL THAT

$$\text{ERR}_{\text{EXT}} = \mathcal{O}(N^{-m}). \quad \leftarrow \begin{cases} \text{TRAPEZOID } m=2 \\ \text{MIDPOINT } m=2 \\ \text{THREE POINTS } m=4 \end{cases}$$

IN OUR CASE, $m \rightarrow \infty$, THIS ANALYSIS PREDICTS PRECIPITOUS CONVERGENCE — APPARENTLY WITH INFINITE SLOPE!

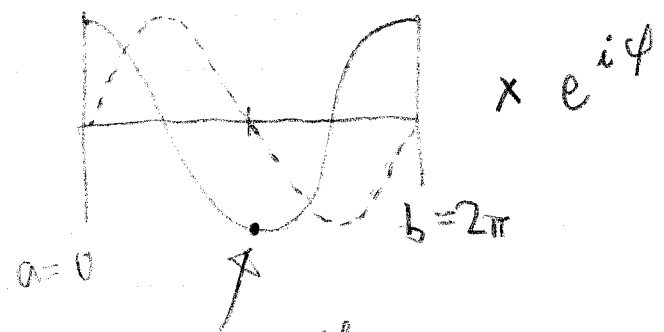
Q: IS THE TRUNCATION ERROR ZERO FOR ALL N?

A: HERE IS A COUNTEREXAMPLE:

$f(x) = e^{i(x+\phi)}$ ← {OUR RESULTS WILL PROVE TO BE INDEPENDENT OF PHASE, ϕ .

LET $a=0, b=2\pi$.

N=1:



$x_1 = \pi, f_1 = -e^{i\phi}, h = 2\pi$

$\int_a^b f dx \approx h \sum f_n = -2\pi e^{i\phi} \rightarrow$ WRONG!

N=2:

$h = \pi$
 $x_n = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}$

$f_n = \{ i, -i \} e^{i\phi}$

$\int_a^b f dx \approx h \sum f_n = 0 \checkmark$

N=3:

$h = \frac{2}{3} \pi$

$x_n = \left\{ \frac{\pi}{3}, \pi, \frac{5\pi}{3} \right\} = \{ 60^\circ, 180^\circ, 300^\circ \}$

$f_n = \left\{ \frac{1+\sqrt{3}i}{2}, -1, \frac{1-\sqrt{3}i}{2} \right\} e^{i\phi}$

$\int_a^b f dx = h \sum f_n = 0 \checkmark$

SO, WHAT HAPPENED?

SUDDEN ONSET OF CONVERGENCE AT N = 2.

Q: IN TERMS OF h, WHAT WAS THE FREQUENCY OF f(x)?

A: $\nu = 1 \text{ cycle} / 2h$

Q: DOES THIS LOOK FAMILIAR?

A: IT'S THE NYQUIST FREQUENCY.

EXPLANATION

RECALL FROM SHANNON'S SAMPLING THEOREM THAT A BAND-LIMITED SIGNAL CAN BE RECONSTRUCTED EXACTLY FROM THE SAMPLED DATA.

BAND LIMITED \iff HIGHEST FREQUENCY $\leq \frac{1}{2h}$.

SO FOR EXAMPLE,

$$\tilde{f}_1 \equiv \sum_{n=1}^N f_n$$

$$\Rightarrow \int_a^b f(x) \approx h \tilde{f}_1 \quad (\text{EXTENDED MIDPOINT RULE})$$

IF f(x) IS BAND LIMITED AT N SAMPLES, THEN WE ALREADY KNOW THE DFT FOR N' >> N SAMPLES:

E.G. $\tilde{f}'_n = \frac{N}{N'} \tilde{f}_n, \quad n < \frac{N'}{2} + 1$

SO THE EXTENDED MIDPOINT RULE FOR N' STEPS IS

$$\int_a^b f(x) = h' \tilde{f}'_1 = h' \frac{N}{N'} \tilde{f}_1 = h \tilde{f}_1$$

REMARK: N-STEP RESULT EXACTLY EQUALS THE N' $\rightarrow \infty$ LIMIT OF THE EXT. MIDPOINT RULE.

OMIT

PRACTICAL APPLICATION NOTES

- SUDDEN ONSET OF CONVERGENCE IS PREDICTED UPON THE SIGNAL BEING BAND-LIMITED.

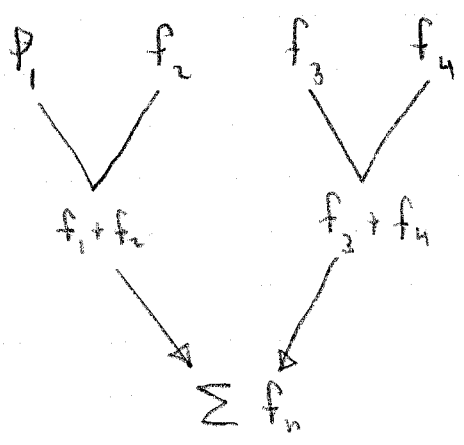
→ LOOK AT WHAT HAPPENS WHEN WE RELAX THIS ASSUMPTION.

- IMPLEMENTATION OF $\sum f_n$ IS SUBJECT TO ROUND OFF ERROR.

→ FFT DOES SUMMATION HIERARCHICALLY BY FACTORING N. THIS MINIMIZES ROUND OFF ERROR.

Q: DOES THAT MAKE SENSE?

A: HIERARCHICAL SUMMATION MEANS:



THIS AVOIDS THE TRAP OF ADDING A BIG NUMBER TO A SMALL ONE.

EXAMPLES...

(16)

Some functions that would be fun to integrate on $(0, \pi]$.

$$\blacksquare y = \left(x \left(1 - \frac{x}{\pi}\right)\right)^2$$

Analytic except at the endpoint, where it is C^2 . Integral can be done analytically.

$$\blacksquare y = 1 + \cos(\pi \cos x)$$

Analytic, infinite Fourier series, but the integral can't be done analytically.

$$\blacksquare y = \sin^4 x$$

Analytic, finite Fourier series, and the integral can be done analytically.

$$\blacksquare y = \exp(-\cos 2x)$$

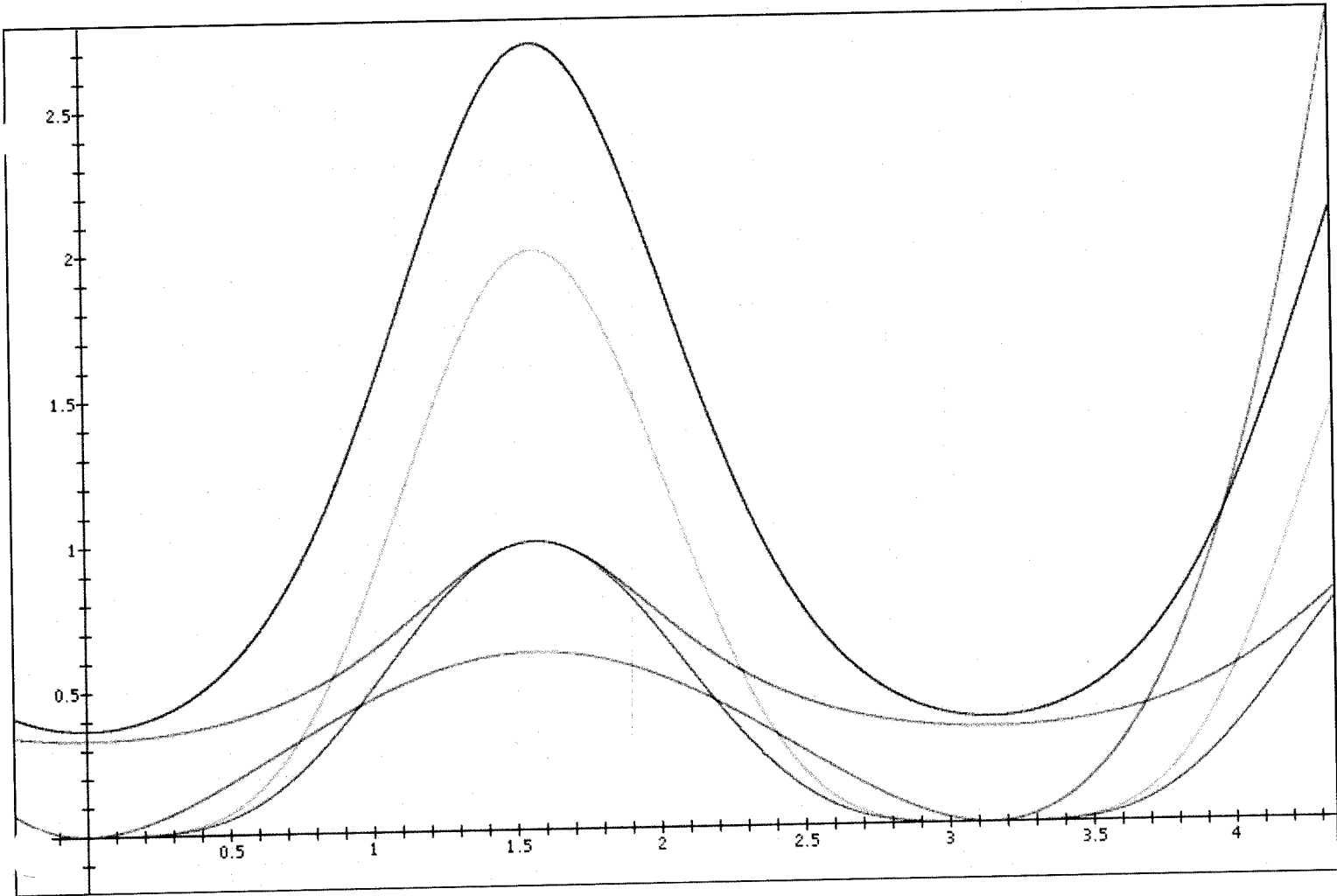
Analytic, infinite Fourier series, but the integral cannot be done analytically

$$\blacksquare y = \frac{1}{2 + \cos 2x}$$

Analytic, infinite Fourier series, integral can be done analytically.

$$\int \frac{1}{\cos(2x) + 2} dx =$$

$$\frac{\tan^{-1}\left(\frac{\tan(x)}{\sqrt{3}}\right)}{\sqrt{3}}$$



This file was created by Graphing Calculator 3.5.
Visit [Pacific Tech](#) to download the helper application to view and edit [these equations](#) live.

```

% A function with a period of pi that can be integrated
% analytically, and has an infinite Fourier series. The
% function and its first and second derivatives are
% continuous at 0 and pi.
% It is normalized so that the integral from 0 to pi is 1.
function result = periodic3(x)
normalization = 30/(pi^3);
result = normalization * ( x * ( 1 - x./pi ) ).^2;

```

1 . periodic3.m . 2009-02-06 12:35 . Charles Kankelborg

% Answer: Consider expanding the fraction as a power series in cos(2x)....
 % Question: How do I know the Fourier series is infinite?

```

% A function with a period of pi that can be integrated
% analytically, and has an infinite Fourier series.
% It is normalized so that the integral from 0 to pi is 1.
function result = periodic2(x)
normalization = sqrt(3)/(pi);
result = normalization ./ ( 2 + cos(2.*x) );

```

1 . periodic2.m . 2009-02-06 12:36 . Charles Kankelborg

```

% A function with a period of pi that has a finite Fourier series.
% It is normalized so that the integral from 0 to pi is 1.
function result = periodic1(x)
normalization = 8/(3*pi);
result = normalization * (sin(x)).^4;

```

1 . periodic1.m . 2009-02-06 12:36 . Charles Kankelborg

```

% FFT integration of a periodic function
function result = fnt(integrand, a, b, N)
% a, b are the limits of integration.
% N is the number of abscissas.
dx = (b-a)/N;
x = a + (0:(N-1))*dx;
f = integrand(x);
% The following is an inefficient way of summing the values
% of f at all the abscissas. But it has very low roundoff error.
F = fft(f);
result = F(1)*dx;

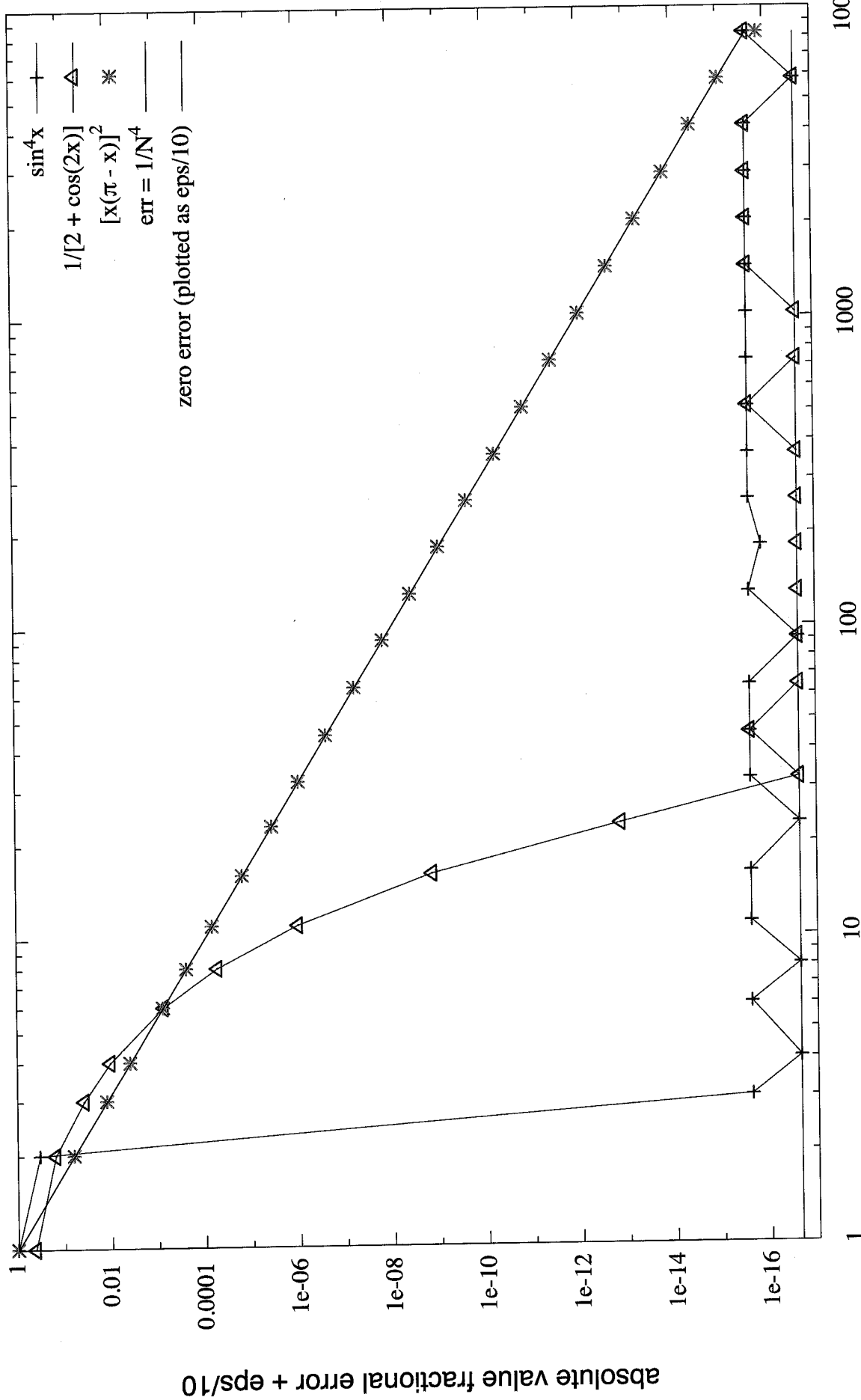
```

```

% Test of FFT quadrature
function test1
mmax = 27;
for m=1:mmax
N(m)=round(2^(m-1)/2);
err1(m) = fnt(@periodic1, 0, pi, N(m)) - 1;
err2(m) = fnt(@periodic2, 0, pi, N(m)) - 1;
err3(m) = fnt(@periodic3, 0, pi, N(m)) - 1;
end
LogLog(N, abs(err1) + eps/10, 'b+', ...
N, abs(err2) + eps/10, 'g-o', ...
N, abs(err3) + eps/10, 'm@', ...
N, N.^(-4), 'k--', ...
[N(1), N(mmax)], [eps/10, eps/10], 'r: ')
title('Convergence of FFT quadrature of periodic integrands', ...
'fontsize', 18)
xlabel('Number of uniformly spaced abscissas', 'fontsize', 16)
ylabel('absolute value fractional error + eps/10', 'fontsize', 16)
legend('sin^4x', ...
'1/|2 + cos(2x)|', ...
'[x(pi - x)]^2', ...
'err = 1/N^4', ...
'zero error (plotted as eps/10)')

```

Convergence of FFT quadrature of periodic integrands



Number of uniformly spaced abscissas