

MULTIDIMENSIONAL INTEGRATION (NR § 4.6)

WE WISH TO COMPUTE:

$$\int_A^B \underbrace{\int_{a(y)}^{b(y)} f(x,y) dx}_{g(y)} dy = \int_A^B g(y) dy$$

SO THE APPROACH WOULD BE:

① CODE THE FUNCTION $G(y) \approx g(y)$.

```
function z = G(y, N)
```

```
z = thint (@(x) f(x,y), a(y), b(y), N);
```

```
end
```

② CODE $f(x,y)$, $a(y)$, $b(y)$

③ function result = main(N)

```
result = thint (@(y) G(y,N), A, B, N)
```

```
end
```

NOW, HOW GOOD IS OUR "result" ?

NOTE: IT WOULD BE INTERESTING TO WRITE A 2D INTEGRATOR THAT WOULD WORK LIKE:

```
result = int2d (@f, A, B, @a, @b, N)
```

ERROR ANALYSIS

THE ERROR OF $g(y)$ WILL BE:

$$\int_a^b f dx = G(y) + \mathcal{O}\left(\frac{1}{N^m}\right),$$

WHERE m IS THE ^{ORDER OF THE} LOWEST DERIVATIVE IN THE TRUNCATION ERROR FOR A GIVEN 1D INTEGRATION ROUTINE. (E.G. TRES HERMINOS, $m=4$).

HOW DOES THE ERROR ACCRUE FOR THE DOUBLE INTEGRAL?

$$\int_A^B \left[\int_a^b f(x,y) dx \right] dy = \int_A^B g(y) dy$$

$$= \int_A^B \left(G(y) + \mathcal{O}\left(\frac{1}{N^m}\right) \right) dy$$

$$= \int_A^B G(y) dy + \mathcal{O}\left(N \cdot h \cdot \frac{1}{N^m}\right)$$

$$= \underbrace{\int_A^B G(y) dy}_{\text{main}(N)} + \mathcal{O}\left(\frac{1}{N^m}\right) + \mathcal{O}\left(\frac{1}{N^m}\right)$$

$$\therefore \int_A^B \int_{a(y)}^{b(y)} f(x,y) dy dy = \text{main}(N) + \mathcal{O}\left(\frac{1}{N^m}\right).$$

GREAT! SCALES THE SAME WITH N AS A SINGLE INTEGRAL! ARE YOU PLEASED?



HERE IS THE PROBLEM —

DOING THE DOUBLE INTEGRAL ENTAILS

$$N = N^2$$

EVALUATIONS OF $f(x,y)$. OVERALL, THEN,

$$\boxed{\text{ERR}_{\text{DOUBLE INT.}} = \mathcal{O}\left(\frac{1}{N^{m/2}}\right)}$$

∴ TRES HERMINOS NOW SCALES LIKE THE TRAPEZOID RULE!

FOR AN M-DIMENSIONAL INTEGRAL,

$$\text{ERR}_m = \mathcal{O}\left(\frac{1}{N^{m/M}}\right)$$

VERY UNFAVORABLE SCALING!

THE ABOVE APPROACH IS SATISFACTORY FOR INTEGRALS OVER 2-3 DIMENSIONS, BUT IMPRACTICAL FOR LARGE DIMENSIONALITY M .

EXAMPLE. 1

WE WISH TO CALCULATE THE RATE (EVENTS PER YEAR) OF BLACK HOLE MERGERS THAT WILL BE OBSERVED BY LISA.

THE LIKELIHOOD OF OBSERVING AN INDIVIDUAL EVENT DEPENDS ON THE PARAMETERS:

SKY POSITION	θ, ϕ
REDSHIFT	z
MASSSES OF BOTH OBJECTS	M, m
INCLINATION OF ORBITAL PLANE TO LINE OF SIGHT	α
CLOCKING ANGLE OF ORBITAL PLANE RELATIVE TO THE OBSERVATORY (\Rightarrow POLARIZATION)	β
(AND PERHAPS OTHERS!)	

THUS THE RATE IS

$$R = \int d\theta \int d\phi \int dz \int dM \int dm \int d\alpha \int d\beta f(\dots)$$

$M=7$. OPTIMISTICALLY, WE HOPE FOR CONVERGENCE $\mathcal{O}(N^{-4})$ ($m=4$) PER DIMENSION. THE CONVERGENCE RATE OF THE WHOLE CALCULATION IS THIS

$$\mathcal{O}(N^{-4/7}).$$

EACH EVALUATION OF f REPRESENTS A VERY EXTENSIVE COMPUTATION. MUST MINIMIZE N !

Q: WHAT MIGHT WE DO TO SPEED THIS UP?

A: SOME IDEAS

- AVOID REDUNDANT CALCULATIONS. DO BLACK HOLE MERGER SOLUTIONS FOR A SUITE OF (M, m) VALUES, AND SIMULATE OBSERVATION OF EACH AT DIFFERENT $\theta, \psi, z, \alpha, \beta$.
- SOME OF OUR DIMENSIONS ARE PERIODIC, SO SHOULD CONVERGE FAST. LUCKY!
- HAVE A GRADUATE STUDENT WRITE THE CODE.
- LOOK FOR SCALING LAWS. CAN THE RESULT FOR (M, m) BE DEDUCED IF WE ALREADY KNOW THE ANSWER FOR $(2M, 2m)$?

COMMENTS

- MUCH CAN BE DONE TO MAKE THIS JOB MORE TRACTABLE (SEE ABOVE).
- CODING $M > 2$ NESTED INTEGRALS IS A LOT OF WORK, THOUGH OF COURSE IT CAN BE DONE.
- SOME INTEGRATION ROUTINES (E.G. OCTAVE'S QUAD, WHICH USES FORTRAN QUADPACK) CAN'T BE CALLED RECURSIVELY (NESTED).
- EVEN LARGER M ARE VERY COMMON. CONSIDER THE LIKELY TOTAL SIGNAL FROM MANY EVENTS, STARTING AT DIFFERENT TIMES, ALL OVER THE SKY, AT ALL z, α, β .