

ORDINARY DIFFERENTIAL EQUATIONS (ODEs) ①

THE TYPICAL FORM FOR AN ODE IS

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t) \quad (1)$$

THIS INCLUDES HIGHER ORDER EQUATIONS, FOR EXAMPLE

$$\ddot{y} = g(t, y, \dot{y})$$

CAN BE REWRITTEN TRIVIALY AS:

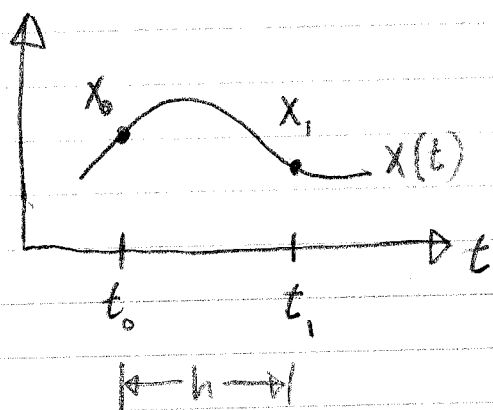
$$\frac{d}{dt} \underbrace{\begin{bmatrix} \dot{y} \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} g(t, y, \dot{y}) \\ \dot{y} \end{bmatrix}}_{\vec{f}(\vec{x}, t)} \quad (2)$$

WHEN THE EQUATION IS IN THE FORM (1),

- \vec{x} IS THE "STATE OF THE SYSTEM." IF $\vec{x}(t_0)$ IS KNOWN, THAT DETERMINES $\vec{x}(t)$.
- \vec{x} IS SAID TO LIVE IN "PHASE SPACE". FOR EXAMPLE, THE PHASE SPACE OF (2) IS EVIDENTLY POSITION (y) AND VELOCITY, OR MOMENTUM (\dot{y}).
- THE INITIAL VALUE PROBLEM IS: GIVEN $\vec{x}(t_0)$, FIND $\vec{x}(t)$. WE'LL HANDLE THIS FIRST.
- THE BOUNDARY VALUE PROBLEM INVOLVES PARTIAL SPECIFICATION OF \vec{x} AT 2 (OR MORE) PLACES, WITH THE SAME TOTAL INFORMATION CONTENT AS $\vec{x}(t_0)$.

(2)

THE INITIAL VALUE PROBLEM — EXPANSION OF A SINGLE STEP.



$$\dot{x} = f(x, t)$$

$$x(t_0) = x_0 \text{ IS GIVEN.}$$

$$\text{WANT } x_1 = x(t_0 + h)$$

TAYLOR SERIES:

$$x_1 = x_0 + \dot{x}_0 h + \frac{1}{2} \ddot{x}_0 h^2 + \frac{1}{6} x_0^{(3)} h^3 + \mathcal{O}(h^4)$$

NOW, OBSERVE THAT

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt} \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + f \frac{\partial}{\partial x}$$

SO USING $\dot{x}_0 = f$,

$$\ddot{x}_0 = \frac{df}{dt} = f_t + f f_x,$$

$$\begin{aligned} x_0^{(3)} &= \frac{d\ddot{x}}{dt} = f_{tt} + f_t f_x + f f_{xt} + f(f_{xt} + f_x^2 + f f_{xx}) \\ &= f_{tt} + f_x f_t + 2f f_{xt} + f f_x^2 + f^2 f_{xx}. \end{aligned}$$

COUPLE THINGS TO NOTE:

① I HAVE WRITTEN f, x AS SCALARS, BUT THEY CAN BE VECTORS. THAT WOULDN'T CHANGE ANYTHING.

② f, f_x, f_t , ETC. ARE ALL EVALUATED AT (x_0, t_0) .

③ $x_0^{(4)}$, NEEDED TO DERIVE RK4, HAS 11 TERMS.

3

EULER'S METHOD — JUST TAKE 1ST 2 TERMS:

$$x(t+h) \approx x(t) + h\dot{x}(t) = x(t) + hf(x(t), t)$$

EXTENDED TO MULTIPLE STEPS:

$$x_{n+1} = x_n + hf(x_n, t_n) + \mathcal{O}(h^2)$$

WHERE $t_n = t_0 + nh$.

EXAMPLE

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

DO YOU RECOGNIZE THIS AS A PHYSICS EQUATION?

$$\frac{d}{dt} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} -(k/m)x \\ \dot{x} \end{bmatrix}$$

SOLN: $i\omega \tilde{x} = -\tilde{y}$
 $i\omega \tilde{y} = \tilde{x}$

$\Rightarrow \omega = \pm 1$

$\Rightarrow x = \alpha e^{it} + \beta e^{-it}$

$\Rightarrow x = R \cos(t + \phi)$

$y = R \sin(t + \phi)$

(4)

RUNGE - KUTTA

(THIS DERIVATION FOLLOWS CONTE & DE BOOR,
ELEMENTARY NUMERICAL ANALYSIS, MCGRAW-HILL 1990.)

OUR SOLUTION WOULD CONVERGE MUCH FASTER IF WE
WENT TO HIGHER ORDER IN h .

THE TAYLOR EXPANSION INVOLVES LOTS OF PARTIAL
DERIVATIVES OF f , BUT WE CAN GET AROUND THAT
BY FINITE DIFFERENCING.

— 2nd ORDER RUNGE KUTTA RK2 —

ANSATZ: $x_1 = x_0 + a k_1 + b k_2 + \mathcal{O}(h^3)$

WHERE $k_1 = h f$ ← EULER TERM
 $k_2 = h f(x_0 + \alpha h, t_0 + \beta k_1)$

WE NOW TAYLOR EXPAND THIS EXPRESSION FOR x_1 :

$$x_1 = x_0 + a h f + b h \left[\underbrace{f}_{0^{\text{th}} \text{ ORDER}} + \underbrace{\alpha h f_t + \beta k_1 f_x}_{1^{\text{st}} \text{ ORDER}} + \underbrace{\frac{\alpha^2 h^2}{2} f_{tt} + \frac{\beta^2 h_1^2}{2} f_{xx} + \alpha h \beta k_1 f_{xt}}_{2^{\text{nd}} \text{ ORDER}} + \mathcal{O}(h^3) \right]$$

NOTE THAT $f, f_x, f_t, f_{xx}, f_{tt}, f_{xt}$ ARE ALL
EVALUATED AT (x_0, t_0) .

(5)

RK2, CONT.

SIMPLIFYING,

$$x_1 = x_0 + (a+b)hf + bh^2(\alpha f_t + \beta ff_x) + \mathcal{O}(h^3). \quad (3)$$

Q: THIS IS OUR ANSATZ. HOW CAN WE TELL IF IT'S TRUE?
AND HOW DO WE CHOOSE a, b, α, β ?

A: COMPARE WITH OUR ORIGINAL TAYLOR EXPANSION OF
 x_1 (P. 2). WE HAD:

$$x_1 = x_0 + hf + \frac{1}{2}h^2(f_t + ff_x) \quad (4)$$

TO MAKE (4) AGREE WITH (3), WE REQUIRE:

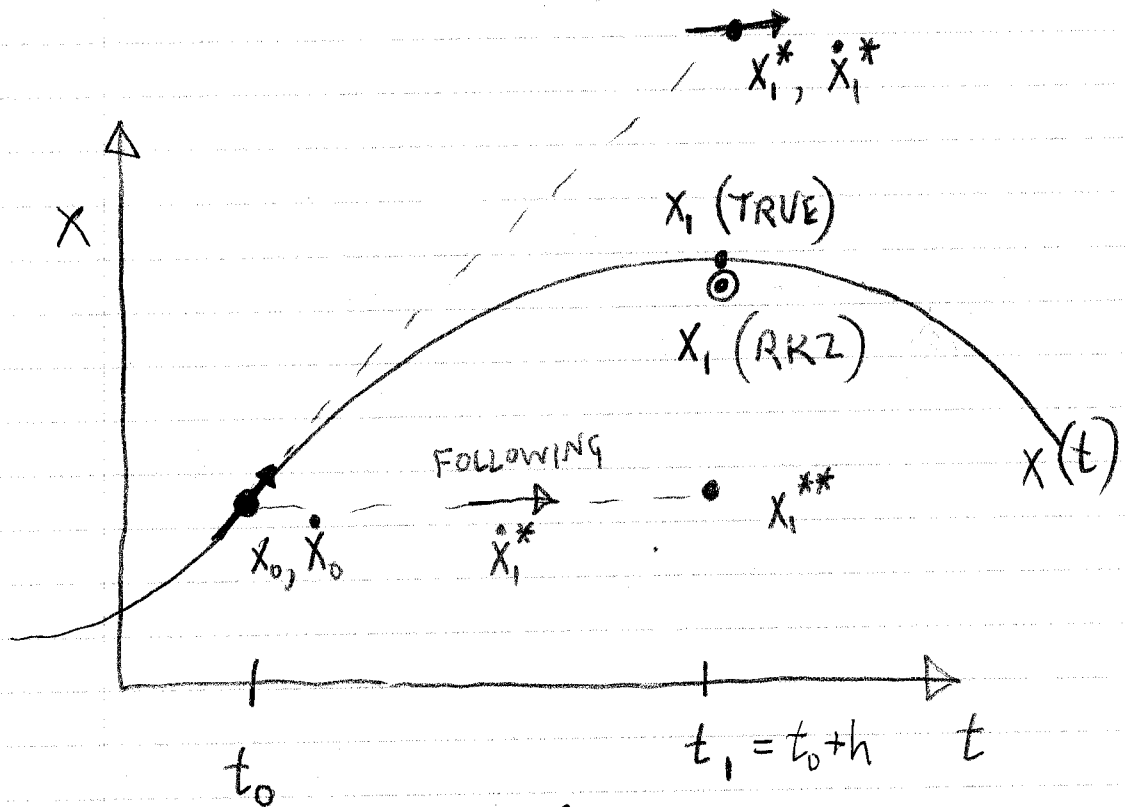
$$a+b = 1, \quad \alpha = \beta, \quad \alpha b = \frac{1}{2}.$$

ANY SCHEME SATISFYING THE ABOVE CONSTRAINTS IS
A "2nd ORDER RUNGE KUTTA". THAT LEAVES ONE
FREE PARAMETER.

RK2, CONT.

ONE POPULAR CHOICE IS:

$$a=b=\frac{1}{2}, \quad \alpha=\beta=1$$



$$x_1^* = x_0 + h f(x_0 + h f(x_0, t_0), t_1)$$

$$\dot{x}_1^* = f(x_1^*, t_1)$$

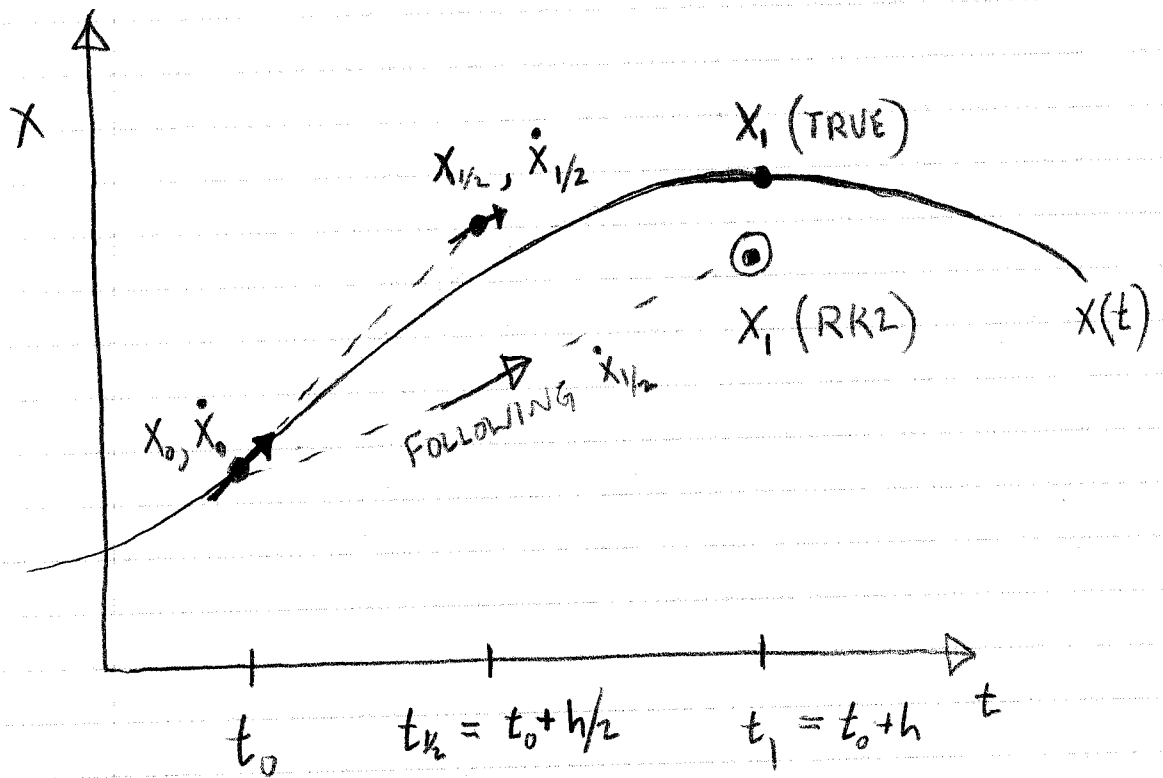
$$x_1^{**} = x_0 + h \dot{x}_1^*$$

$$\begin{aligned} x_1^{\text{RK2}} &= x_0 + \frac{1}{2} h f(x_0, t_0) + \frac{1}{2} h f(x_1^*, t_1) \\ &= \frac{1}{2} (x_1^* + x_1^{**}) \end{aligned}$$

RK2, cont.

MY FAVORITE CHOICE IS:

$$a=0, b=1, \alpha=\beta=\frac{1}{2}$$



$$X_{1/2} = X_0 + \frac{h}{2} \dot{x}_0$$

$$\dot{x}_{1/2} = f(X_{1/2}, t_{1/2})$$

$$X_1^{RK2} = X_0 + h f(X_{1/2}, t_{1/2}) = X_0 + h \dot{x}_{1/2}$$