

STEP DOUBLING

"ADAPTIVE"

GOALS:

- ① DEVELOP MACHINERY TO SUPPORT VARIABLE STEPSIZE, h .
- ② PROVIDE AN ERROR ESTIMATE ON THE FLY.
- ③ PARLAY AN m^{th} -ORDER ROUTINE INTO $m+1^{\text{st}}$ -ORDER.

I WILL DO IT IN A CHEESY WAY. MY APPROACH WILL BE EFFECTIVE, AND HOPEFULLY CLEAR, BUT FAR FROM OPTIMAL.

PLAN:

- STEP DOUBLING FOR m^{th} -ORDER RK
- EXAMPLE: DOUBLED EULER (RK12)
- A VARIABLE-STEP SOLVER, WITH EXAMPLES (I HOPE!)

GENERALIZED STEP DOUBLING

WE WILL TREAT C AS CONSTANT. IS THIS TRUE? DOES IT MATTER?

GIVEN AN m -th ORDER RK,

$$X_1 = \overbrace{X_0 + R_m(f, X_0, t_0, h)}^{X_1^*} + Ch^{m+1} + \mathcal{O}(h^{m+2}),$$

USE TWO HALF-STEPS TO FORM ANOTHER ESTIMATE OF X_1 :

$$X_{1/2} = \overbrace{X_0 + R_m(f, X_0, t_0, h/2)}^{X_{1/2}^*} + C\left(\frac{h}{2}\right)^{m+1} + \mathcal{O}(h^{m+2})$$

$$X_1 = X_{1/2} + R_m(f, X_{1/2}, t_{1/2}, h/2) + C\left(\frac{h}{2}\right)^{m+1} + \mathcal{O}(h^{m+2})$$

$$= \left[X_{1/2}^* + C\left(\frac{h}{2}\right)^{m+1} \right]^* + R_m(f, X_{1/2}^*, t_{1/2}, h/2)^*$$

$$+ C\left(\frac{h}{2}\right)^{m+1} + \mathcal{O}(h^{m+2}) \quad \underbrace{Ch^{m+1} \left(\frac{1}{2^m}\right)}_{C\left(\frac{h}{2}\right)^{m+1}}$$

$$= \underbrace{X_{1/2}^* + R_m(f, X_{1/2}^*, t_{1/2}, h/2)}_{X_1^{**}, \text{ SECOND ESTIMATE OF } X_1} + 2C\left(\frac{h}{2}\right)^{m+1} + \mathcal{O}(h^{m+2})$$

NOW, LOOK AT THE DIFF. BETWEEN OUR 2 ESTIMATES OF X_1 :

$$X_1^{**} - X_1^* = Ch^{m+1} \left[1 - \frac{1}{2^m} \right] = Ch^{m+1} \left(\frac{2^m - 1}{2^m} \right)$$

FROM THIS WE INFER TRUNCATION ERROR OF OUR SECOND ESTIMATE:

$$\boxed{\varepsilon \equiv \frac{1}{2^m} Ch^{m+1} = (X_1^{**} - X_1^*) / (2^m - 1)}$$

... AND USE THIS TO ELIMINATE $m+1$ ST ORDER TRUNCATION ERROR:

$$\boxed{X_1 = X_1^{**} + \varepsilon + \mathcal{O}(h^{m+2})} = X_1^{**} \left(\frac{2^m}{2^m - 1} \right) - X_1^* \left(\frac{1}{2^m - 1} \right)$$

★ BOTH THESE TERMS ALSO CONTRIBUTE TO THE $\mathcal{O}(h^{m+2})$.

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SIMPLE EXAMPLE: DOUBLED EULER (RK2)

$$X_1^* = X_0 + h f(X_0, t_0), \quad X_1 = X_1^* + Ch^2 + \mathcal{O}(h^3)$$

$$X_{1/2}^* = X_0 + \frac{h}{2} f(X_0, t_0)$$

$$X_1^{**} = X_{1/2}^* + \frac{h}{2} f(X_{1/2}^*, t_{1/2})$$

 \uparrow
 $m=1$

$$X_1 = X_1^{**} \left(\frac{2}{2-1} \right) - X_1^* \left(\frac{1}{2-1} \right) + \mathcal{O}(h^3)$$

$$= 2 \left(X_{1/2}^* + \frac{h}{2} f(X_{1/2}^*, t_{1/2}) \right) - \left(X_0 + h f(X_0, t_0) \right)$$

$$X_1 = X_0 + h f(X_{1/2}^*, t_{1/2}) + \mathcal{O}(h^3)$$

THIS IS JUST RK2!

- DOUBLE AS NEEDED UNTIL $|\epsilon| <$ SPECIFIED TOLERANCE.
- NOTE THAT THE ERROR ESTIMATE IS VERY CONSERVATIVE:

$$\epsilon = \frac{X_1^{**} - X_1^*}{2} = \mathcal{O}(h^2)$$

- PROFESSIONAL GRADE SOLUTIONS (RK45) ARRANGE THE RK_m SO THAT X_1^* USES f EVALUATED IN m OF THE SAME LOCATIONS AS X_1^{**} . SO THERE ARE NO UNNECESSARY FUNCTION CALLS. FURTHERMORE, SUCCESSIVE DOUBLINGS INHERIT PREVIOUS RESULTS, SO f IS NEVER UNNECESSARILY EVALUATED AT THE SAME x, t TWICE.