

EXAMPLE: GRAVITATIONAL N-BODY PROBLEM (2D)

THIS IS A CHALLENGING PROBLEM FOR SEVERAL REASONS

- $[N \times 2 \times M]$ COUPLED, NONLINEAR ODEs.
BODIES \uparrow \uparrow DIMENSIONS
- N SINGULARITIES IN THE GRAVITATIONAL POTENTIAL.
- $\mathcal{O}(N^2)$ FORCE TERMS.

IT'S ALSO AN IMPORTANT PROBLEM:

- SPACECRAFT ORBITAL MECHANICS
- CELESTIAL MECHANICS / EPHEMERIS
- MODELING STAR CLUSTERS, GALAXIES, ETC.

N MUTUALLY-GRAVITATING BODIES IN M DIMENSIONS

$$\boxed{\frac{d\vec{x}_k}{dt} = \vec{v}_k}, \quad \vec{x}_k, \vec{v}_k \in \mathbb{R}^M$$

$k = 1, 2, \dots, N$

$$m_k \frac{d\vec{v}_k}{dt} = - \sum_{l=1}^N \frac{m_k m_l \vec{r}_{kl}}{r_{kl}^3}$$

$$\boxed{\frac{d\vec{v}_k}{dt} = - \sum_{l=1}^N \frac{m_l (\vec{x}_k - \vec{x}_l)}{|\vec{x}_k - \vec{x}_l|^3}}$$

NUMERICAL ASPECTS OF THE N-BODY PROBLEM

- OFTEN FAILS WITH STEPSIZE UNDERFLOW (EXCEEDS MAXIMUM NUMBER OF RECURSIONS)
- HARDER TO RUN SUCCESSFULLY AS N INCREASES.
- "COLLISIONS" ARE THE HARD PART.

WHY?

- MOMENTUM APPARENTLY CONSERVED (TO WITHIN ROUND OFF ERROR)
- ENERGY NOT CONSERVED (TRUNCATION ERROR)

→ EXPRESSIONS FOR TRUNCATION ERROR ARE LINEAR IN THE DERIVATIVES:

$$x_i = x_i^* + K f^{(m)} h^m$$

- TRUNCATION ERROR IN THE FORCE ($d\vec{P}_k/dt$) IS SUM OF TRUNCATION ERRORS ASSOCIATED WITH ALL THE OTHER MASSES, $l \neq k$.
- TRUNCATION ERROR IN $d\vec{P}_k/dt$ DUE TO m_l IS EQUAL AND OPPOSITE TRUNCATION ERROR IN $d\vec{P}_l/dt$ DUE TO m_k .
- ∴ EACH MOMENTUM ERROR IS BALANCED

$$\Delta P_{kl} = -\Delta P_{lk} \Rightarrow \sum \Delta P = 0$$

* PHYSICAL INSIGHT (CONSERVED QUANTITIES) HELPS OUR UNDERSTANDING.

* LEARN THE LIMITATIONS OF THE NUMERICS BEFORE TRYING TO EXTRACT PHYSICS INSIGHT FROM A MODEL.

* JUST BECAUSE IT COMES OUT OF THE COMPUTER DOESN'T MEAN IT'S RIGHT.

PHYSICS OF THE N-BODY PROBLEM

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"THE PURPOSE OF COMPUTING IS INSIGHT, NOT NUMBERS." —HAMMING

- PREDICTABLE, KEPLERIAN ORBITS FOR $N=2$.
- MORE COMPLICATED FOR $N \geq 3$.
 - "HAIRBALL" TRAJECTORIES
 - OFTEN, THE MOTION "ORGANIZES" ITSELF IN SOME WAY:
 - PAIRING UP
 - EJECTING MEMBERS (GLOBULAR CLUSTERS "EVAPORATE")

Q: IN WHAT SENSE ARE THESE TRAJECTORIES TIME-REVERSIBLE?

- FORMALLY, THE EOM ARE SYMMETRIC UNDER TIME REVERSAL.
- COULD YOU DISTINGUISH THE START FROM THE END IF I HADN'T MARKED IT?

Q: IS THE MOTION CHAOTIC?

- A HALLMARK OF CHAOS IS SENSITIVITY TO INITIAL CONDITIONS.
- EACH NEAR-COLLISION SERVES AS A DECISION POINT:
SLIGHT $\Delta \vec{x}_k \Rightarrow$ BIG $\Delta \vec{F}_k$.
- EXAMPLE (NEXT PAGE)

THE MOST VALUABLE THING YOU CAN DO WITH A NUMERICAL SIMULATION IS TO GIVE IT A CLEAR, PHYSICAL EXPLANATION.

NO SIMULATION IS SO COMPLEX THAT IT CANNOT BE UNDERSTOOD BY A GOOD PHYSICIST.

LSODE — LIVERMORE SOLVER FOR ORDINARY DIFF ER'S.

- OCTAVE'S IMPLEMENTATION EXPECTS STATE VECTORS TO BE COLUMN VECTORS, SO I WROTE lsode2.m
- CONVENIENTLY ALLOWS YOU TO SPECIFY TIME POINTS FOR WHICH THE TRAJECTORY SHOULD BE RETURNED. (UNLIKE NR!)
- I'VE PLACED NASA RP-1327 (126 PP!) ON THE WEBSITE.
- BY DEFAULT, LSODE USES ADAMS-MOULTON METHOD OF ORDERS 1-12. THIS IS A PREDICTOR-CORRECTOR ALGORITHM. SEE NR CH. 16

WHAT DO WE LEARN FROM APPLYING LSODE TO THE N-BODY PROBLEM?

FOR THE SETTINGS IN NBODY_TEST3, LSODE PERFORMS BETTER THAN CCK.

- LSODE IS SOMEWHAT (FACTOR OF A FEW) MORE ACCURATE THAN CCK.
- CCK RUNS $\sim 1-100$ TIMES AS LONG AS LSODE. (REMEMBER THAT LSODE IS FORTRAN, CCK IS INTERPRETED)
- LSODE NEVER* FAILS; CCK RUNS VERY DEEP IN RECURSION, AND IT IS NOT CLEAR WHETHER IT WOULD COMPLETE IF ALLOWED TO DO SO.
- WHEN BOTH METHODS ARE USED WITH THE SAME INITIAL CONDITIONS, THE RESULTING TRAJECTORIES ARE INDISTINGUISHABLE.

→ WE CAN BE CONFIDENT THAT THE TRAJECTORIES OBTAINED USING CCK OR LSODE ARE PHYSICALLY REALISTIC. THEY ARE INDEPENDENT OF METHOD.

→ LSODE OFFERS IMPROVED SPEED & ACCURACY. IT IS A PROFESSIONAL QUALITY TOOL.

*NOT YET, ANYWAY.