

~~NEW~~ "TRICK SHOOTING"

I. HIGH-DIMENSIONAL (> 2) SYSTEMS OF ODEs:

NR § 17.1 shoot()

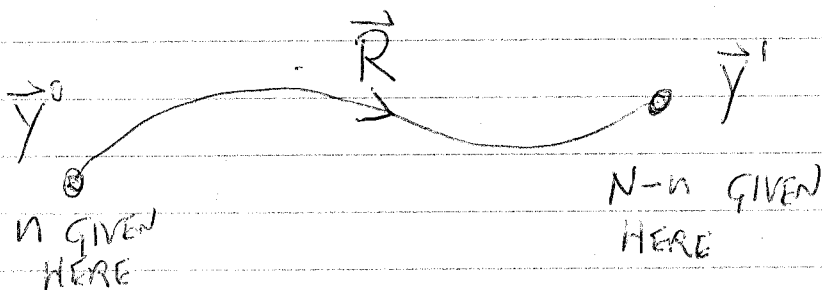
NR § 9.6 NEWTON'S METHOD, `minwt()`
FOR SYSTEMS OF NONLINEAR EQ'S

THE SHOOTING PROBLEM IN N DIMENSIONS:

$$\frac{d}{dt} \vec{y} = \vec{F}(\vec{y}, t), \quad \vec{y} \in \mathbb{R}^N$$

INTEGRATE FROM $\vec{y}^0 = \vec{y}(t_0)$ TO $\vec{y}^1 = \vec{y}(t_1)$,
GIVEN BOUNDARY CONDITIONS:

$$y_1^0, y_2^0, y_3^0, \dots, y_n^0; y_{n+1}^1, y_{n+2}^1, \dots, y_N^1$$



WE HAVE A PROCEDURE \vec{R} THAT INTEGRATES:

$$\vec{R}(\vec{y}^0) = \vec{y}^1$$

CONSIDER $N-n$ COMPONENTS TO SOLVE FOR $t=0$ UNKNOWNNS:

$$R_i(\vec{y}^0) = y_n^1, \quad n+1 \leq i \leq N$$

$N-n$ EQS, $N-n$ UNKNOWNNS.

REWRITE AS

$$G_i (Y_{n+1}^0 \dots Y_N^0) \equiv R_n(\vec{Y}^0) - Y_i^1 = 0$$

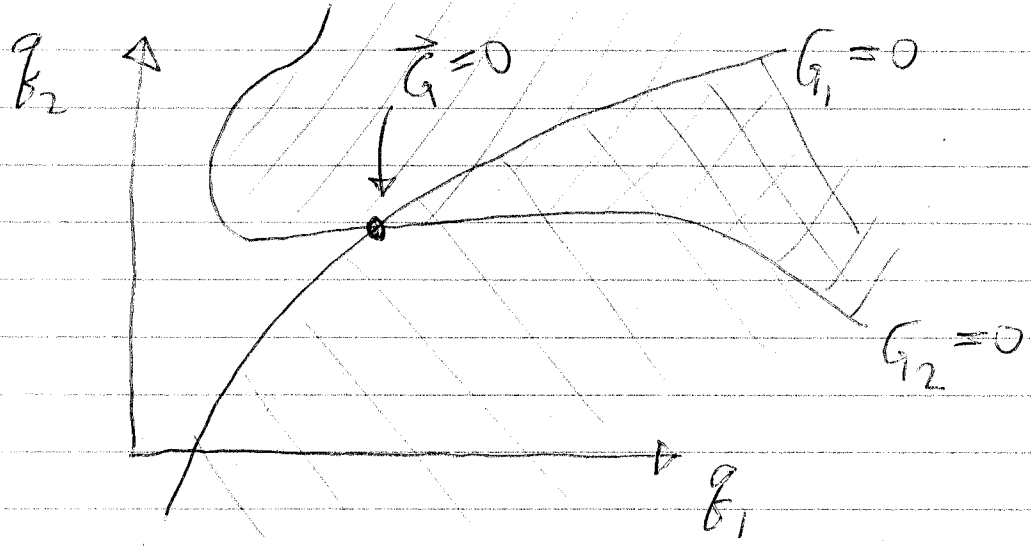
$$\star \boxed{\vec{G}(\vec{q}) = 0}, \quad \vec{G}, \vec{q} \in \mathbb{R}^M, \quad M \equiv N - m$$


⇒ MULTIDIM. ROOT FINDING.

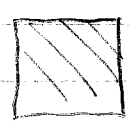
[AND ~~WHEN~~ WHEN WE'RE DONE, WE RUN \vec{R} ONCE TO INTEGRATE FROM \vec{Y}_n^0 ~~TO~~ (KNOWN) TO \vec{Y}_n^1 .]

THE HARD PART IS SOLVING \star .

CAN WE BRACKET THE ROOT? CONSIDER $M=2$:



 = $G_2 > 0$

 = $G_1 > 0$

WE CAN ONLY SAMPLE INDIVIDUAL POINTS, AND NO NUMBER OF POINTS CAN BE USED TO ~~OR~~ ENCLOSE A (NULL) POINT IN 2D.

NO. CANNOT BRACKET.

SINCE WE CANNOT BRACKET ROOTS IN \mathbb{R}^m
FOR $m > 1$ DIMENSIONS,

~~X~~ ANALOGUE TO BISECTION

~~A~~ GUARANTEE OF CONVERGENCE

ALL WE CAN DO IS USE SLOPES TO SEARCH FOR
ROOTS. RECALL NEWTON'S METHOD:

$$q^* = q - \frac{G(q)}{G'(q)}$$

WE SEEK A MULTI-DIMENSIONAL ANALOGUE...

WANT TO SATISFY

$$\vec{G}(\vec{q}^*) = 0 \quad \star$$

SUPPOSE $\vec{q}^* = \vec{q} + \vec{\delta}_q$

$$\vec{G}(\vec{q}^*) = \vec{G}(\vec{q}) + \sum_i \frac{\partial \vec{G}}{\partial q_i} \bigg|_{\vec{q}} \delta q_i + \mathcal{O}(\|\vec{\delta}_q\|^2)$$

OR IN COMPONENT FORM,

$$G_j(\vec{q}^*) \approx \sum_i \frac{\partial G_j}{\partial q_i} \bigg|_{\vec{q}} \delta q_i + G_j(\vec{q})$$

JACOBIAN MATRIX J

IF WE WANT TO SATISFY \star ,

$$J \vec{\delta}_q + \vec{G}(\vec{q}) = 0$$

$$\Rightarrow \delta \vec{q} = -J^{-1} \vec{G}(\vec{q})$$

$$\Rightarrow \boxed{\vec{q}^* = \vec{q} - J^{-1} \vec{G}(\vec{q})}$$

THIS IS NEWTON'S METHOD IN N-DIMENSIONS.

NR § 9.6, mnewt()

THIS IS ALL WE NEED TO SOLVE THE GENERAL 2-POINT B.V. PROBLEM FOR AN N-DIMENSIONAL ODE.

Q: WHAT IS THE RELATIONSHIP BETWEEN
N — SIZE OF PHASE SPACE
M — # DIMENSIONS OF $\vec{G}(\vec{q})$

A: M IS AT MOST N-1. BUT IF WE'RE FREE TO CHOOSE THE DIRECTION OF INTEGRATION,

$$M \leq \frac{N}{2}.$$

NONLINEAR EQUATION SOLVERS:

OCTAVE f solve (MINPACK hybrid)*
MATLAB f solve (MULTIPLE ALGORITHMS)

ANOTHER STRATEGY FOR SOLVING

$$\vec{G}(\vec{q}) = 0$$

IS TO LET $f(\vec{q}) \equiv |\vec{G}|^2$.

THIS IS NOW RECAST AS A PROBLEM OF MINIMIZING
A SCALAR FUNCTION OF SEVERAL VARIABLES.

MINIMIZATION IS A TOPIC OF BROAD INTEREST, WITH MANY APPLICATIONS.

"One of the principal ways that the methods described so far can fail is if J (in Newton's method) or B in (Broyden's method) becomes singular or nearly singular, so that δx cannot be determined. If you are lucky, this situation will not occur very often in practice." (NR 2ed., sec. 9.7). Levenberg-Marquardt methods are a robust approach to this problem. See, for example, <https://www.lehigh.edu/~wes1/apci/24feb00-s.pdf>

* POWELL HYBRID METHOD, A NEWTON-BROYDEN SOLVER.