

A SIMPLIFIED MODEL OF A SOLAR CORONAL LOOP*

(1)

CONSTANT PRESSURE:

$$P = 2n_e kT = 2n_{e0} kT_0$$

$$n_e = n_{e0} T_0 T^{-1}$$

ENERGY BALANCE:

$$\frac{dE}{dt} = 0 = -\nabla \cdot \vec{F}_c + E_H - n_e^2 \Lambda(T),$$

↑ THERMAL CONDUCTION
↑ HEATING
↑ RADIATION

WHERE $\vec{F}_c = -k T^{5/2} \frac{dT}{ds} \hat{s}$

$$\Rightarrow -\nabla \cdot \vec{F}_c = \frac{5}{2} k T^{3/2} T'^2 + k T^{5/2} T''$$

$$\Rightarrow 0 = k T^{5/2} T'' + \frac{5}{2} k T^{3/2} T'^2 + E_H - n_{e0}^2 T_0^2 T^{-2} \Lambda(T)$$

$$\frac{d}{ds} T' = \frac{n_{e0}^2 T_0^2}{k T^{9/2}} \Lambda(T) - \frac{5}{2} \frac{T'^2}{T} - \frac{E_H}{k_0 T^{5/2}}$$

$$\frac{d}{ds} T = T' \quad K = 10^{-6} \text{ (SPITZER)}$$

$$\Lambda(T) = 3 \times 10^{-21} T^{-1/2}, \quad 10^{4.5} < T < 10^7$$

(CF. MARTENS, KINKELBORG & BERGER 2006)

* SEE, E.G., ROSNER, TUCKER & VAIANA 1978 (APJ 233:987-997)

- ALL UNITS cgs.

- CONSTANT PRESSURE (I.E. GRAVITY IS NEGLECTED)

- CONFINED TO A MAGNETIC FLUX TUBE OF CONSTANT WIDTH.

- E_H COULD BE A FUNCTION OF POSITION, BUT WE ASSUME IT IS CONSTANT.

octave:1> looptest2

Options for LSODE include:

keyword	value
-----	-----
absolute tolerance	1.49012e-08
relative tolerance	1.49012e-08
integration method	stiff
initial step size	-1
maximum order	-1
maximum step size	-1
minimum step size	0
step limit	100000

P0 = 5.0000e+16
T0 = 1000000
ne0 = 5.0000e+10
EH = 0.0020000
Npoints = 1000
s1 = 10000
sN = 5.0000e+09
Tchromosphere = 3.1623e+04
gradTchromosphere = 0
FCchromosphere = 0

LENGTH TURNS OUT TO BE $L \sim 4 \times 10^9 \text{ cm}$
 $T_{\text{top}} \sim 2.7 \times 10^6 \text{ K}$

LOOP MODEL SHOOTING

ADJUSTABLE PARAMETERS:

PRESSURE $p = n_{e0} T_0$, ASSUMED CONSTANT.

HEATING E_H

LOWER BC: $\nabla T (F_c)$ AT CHROMOSPHERE
(THESE ARE NORMALLY SET TO ZERO)

SHOOT FOR DESIRED VALUES OF:

LOOP HALF-LENGTH L
TEMPERATURE T_{TOP}

WE NEED INITIAL GUESSES OF P AND E_H FOR SHOOTING. WHERE SHALL WE START?

→ NOTE HOW EASY IT IS TO CRASH LSODE IF THE LENGTH COMES UP TOO SHORT ($2L < S_N$)!

... WE WANT A GUESS THAT WILL GET US CLOSE TO THE RIGHT ANSWER. USE LOOP SCALING LAWS DEVELOPED BY ROSNER ET AL (1978):

$$T_{top} \propto (pL)^{1/3} \Rightarrow p \propto T^3/L$$

$$E_H \propto p^{7/6} L^{-5/6}$$

THESE APPROXIMATE RELATIONS CAN BE SCALED BY PLUGGING IN RESULTS FROM looptest2:

$$\frac{T^3 L}{p} = \frac{(2.7 \times 10^6)^3 \cdot 4 \times 10^9}{5 \times 10^6} = 1.6 \times 10^{12}$$

$$\Rightarrow p \approx \frac{T^3 L}{1.6 \times 10^{12}}$$

$$\frac{p^{7/6} L^{-5/6}}{E_H} = \frac{(5 \times 10^{16})^{7/6} (4 \times 10^9)^{-5/6}}{2 \times 10^{-3}} = 1.5 \times 10^{14}$$

$$\Rightarrow E_H \approx \frac{p^{7/6} L^{-5/6}}{1.5 \times 10^{14}}$$


```
octave:12> shootloop(1e9, 1e6, 0)
desired_temperature = 1000000
desired_length = 1.0000e+09
kappa = 1.0000e-06
Tchromosphere = 3.1623e+04
gradTchromosphere = -0
Pguess = 6.2500e+14
EHguess = 3.8527e-05
prams_guess =
```

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0.62500 0.38527
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```
badness = 0.36637
badness_initial_guess = 0.36637
badness = 0.36637
badness = 0.14378
badness = 0.61028
badness = 0.14474
badness = 0.12253
badness = 0.14290
badness = 0.10709
badness = 0.11196
badness = 0.22405
badness = 0.16988
badness = 0.12635
badness = 0.13983
badness = 0.10288
badness = 0.094426
badness = 0.076797
badness = 0.066079
badness = 0.25507
badness = 0.095554
badness = 0.077809
badness = 0.077828
badness = 0.072659
badness = 0.058713
badness = 0.050698
badness = 0.056530
badness = 0.039212
badness = 0.031214
badness = 0.028731
badness = 0.026661
badness = 0.0087833
badness = 0.0021184
badness = 0.020595
badness = 0.013599
badness = 0.0020181
badness = 0.018029
badness = 0.048487
badness = 0.020595
badness = 0.0083869
badness = 0.0041228
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badness = 8.2791e-06
badness = 0.0028464
badness = 0.013889
badness = 0.0041988
badness = 3.4953e-04
badness = 0.0012864
badness = 0.0031750
badness = 0.0014567
badness = 7.8293e-04
badness = 7.1515e-04
badness = 4.6662e-04
badness = 1.4031e-04
badness = 3.5935e-05
badness = 6.3243e-05
badness = 1.4387e-04
badness = 2.0675e-05
badness = 4.8791e-06
badness = 1.1104e-05
badness = 1.6927e-05
badness = 3.2937e-06
badness = 1.7380e-06
badness = 6.3290e-08
badness = 5.0438e-06
badness = 1.5498e-06
badness = 2.4752e-07
badness = 3.6330e-07
badness = 2.0977e-07
badness = 8.4113e-07
badness = 3.3338e-07
badness = 1.5309e-07
badness = 6.6919e-08
badness = 2.1871e-09
badness = 1.6230e-08
badness = 7.9426e-08
badness = 2.5509e-08
badness = 1.1895e-08
badness = 5.3749e-09
badness = 2.7973e-08
badness = 1.1773e-08
badness = 8.0281e-09
badness = 4.4134e-09
badness = 1.8601e-09
badness = 2.0924e-09
badness = 1.1815e-08
badness = 5.7767e-09
badness = 2.6911e-09
badness = 3.7056e-09
badness = 1.5351e-09
badness = 1.5568e-09
badness = 3.9804e-10
badness = 4.4403e-11
badness = 2.6734e-09

badness = 1.1338e-09
badness = 3.9857e-10
badness = 5.2367e-10
badness = 1.8596e-10
badness = 1.1888e-10
badness = 7.1319e-11
badness = 2.6348e-10
badness = 9.1609e-11
badness = 5.0822e-11
badness = 1.0224e-11
bestfit_prams =

9.4982 9.1733

badness = 1.0224e-11
badness_bestfit = 1.0224e-11
P0_best = 9.4982e+15
EH_best = 9.1733e-04

1

I INTRODUCED NULLS IN THE CONTEXT OF SHOOTING BECAUSE INTEGRATING UP TO A NULL CAN BE DIFFICULT. NULLS ALSO REVEAL THE CHARACTER OF THE ODE PHASE SPACE. IN THIS CONNECTION, LET'S REVISIT THE CORONAL LOOP MODEL:

(1a) $\frac{dT}{ds} = T'$

(1b) $\frac{dT'}{ds} = \frac{(n_{e0} T_0)^2 \Lambda(T)}{k T^{9/2}} - \frac{5}{2} \frac{T'^2}{T} - \frac{E_H}{k T^{5/2}}$

RADIATION HEATING
 ↓ ↓
 CONDUCTION

WHERE $k = 10^{-6}$

$\Lambda(T) = 3 \times 10^{-21} T^{-1/2}$ } CURVE APPROXIMATION

$10^{4.5} < T < 10^7$

IT WILL BE EASIER TO LOOK AT THESE EQUATIONS IF WE USE MORE PHYSICALLY APPROPRIATE UNITS:

LET $n_{e0} T_0 = 10^{10} \cdot 10^6 \text{ P} = 10^{16} \underline{\text{P}}$

$E_H = 10^{-3} \underline{\text{H}}$

$s = 10^9 \text{ cm } \underline{\underline{t}}$

THE EQ'S THEN BECAME:

$x \equiv T / 10^6 \text{ K}$

(2a) $\frac{dx}{dt} = y$

(2b) $\frac{dy}{dt} = \frac{3 \text{ P}^2}{10} x^{-5} - \frac{5}{2} \frac{y^2}{x} - \text{H} x^{-5/2}$



1.5

(OPTIONAL TANGENT)

THE LOOP SCALING LAWS CAN BE WORKED OUT BY SETTING

CONDUCTION TERMS = RADIATION TERM

HEATING TERM = CONDUCTION PLUS RADIATION

THE GRADIENTS T' ARE THEN WRITTEN T/L .

(2)

NOW LOOK FOR NULL(S):

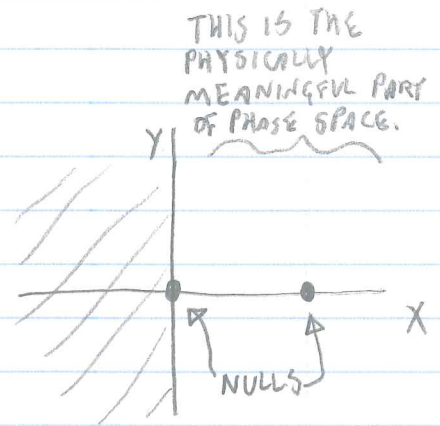
$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad @ \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\Rightarrow y_0 = 0,$$

$$(3) \quad \frac{3P^2}{10} x_0^{-5} - H x_0^{-5/2} = 0$$

$$\Rightarrow x = 0 \text{ (TRIVIAL), OR}$$

$$x_0 = \left(\frac{3P^2}{10H} \right)^{2/5}$$



THIS CORRESPONDS TO A TEMPERATURE AT WHICH HEATING & RADIATION BALANCE.

Q: NOW, WHAT IS THE JACOBIAN? (EASY TO MIX UP THE OFF-DIAG ELEMENTS)

$$J = \begin{bmatrix} \partial \dot{x} / \partial x & \partial \dot{x} / \partial y \\ \partial \dot{y} / \partial x & \partial \dot{y} / \partial y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{3}{2} P^2 x^{-6} + \frac{5}{2} \frac{y^2}{x^2} + \frac{5}{2} H x^{-7/2} & 5 y/x \end{bmatrix}$$

AT THE NULL:

$$J_0 = \begin{bmatrix} 0 & 1 \\ \underbrace{-\frac{3}{2} P^2 \left(\frac{3P^2}{10H} \right)^{-14/5} + \frac{5}{2} H \left(\frac{3P^2}{10H} \right)^{-7/5}}_{-k^2} & 0 \end{bmatrix}$$

LOOKING BACK AT (3), IT IS EVIDENT THAT $k^2 > 0$, I.E. $k \in \mathbb{R}$.

③

THE EIGENVALUES E MUST SATISFY:

$$\begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = E \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{aligned} b &= E a \\ -k^2 a &= E b = E^2 a \end{aligned}$$

$$E^2 = -k^2$$

$$E = \pm i k, \quad k \in \mathbb{R}$$

THE GENERAL SOLUTION IS:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \propto e^{\pm i k t}$$

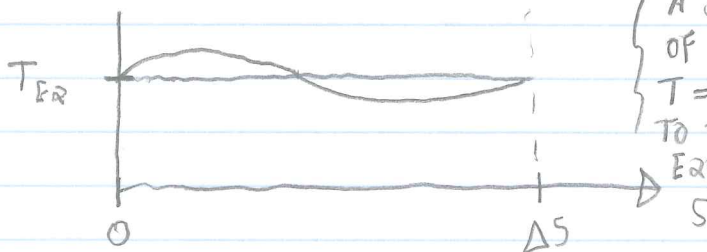
Q: WHAT IS THE PHYSICAL SIGNIFICANCE OF k ?

UNITS: kt IS A PURE NUMBER

t IS A LENGTH (IN UNITS OF 10^9 cm)

$\therefore 1/k$ IS A LENGTH (")

IN PARTICULAR, $kt = 2\pi$, OR $\Delta s = \frac{2\pi}{k} \times 10^9$ cm,
IS THE PERIOD OF A PERTURBATION ON THE EQUILIBRIUM
TEMPERATURE, $T_{ER} = X_0 \times 10^6$ K:



A SINUSOIDAL PERTURBATION
OF PERIOD Δs ABOUT
 $T = T_{ER}$ IS A SOLUTION
TO THE ENERGY BALANCE
EQUATION.

(4)

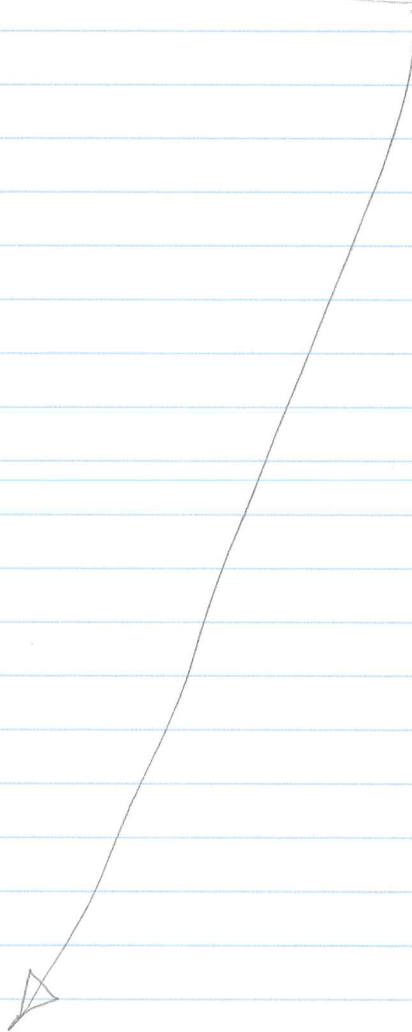
WORK OUT k FOR $P=1, H=1$

$$k^2 = \frac{3}{2} \left(\frac{3}{10}\right)^{-12/5} - \frac{5}{2} \left(\frac{3}{10}\right)^{-7/5} = 13.49$$

$$k = 3.67 \Rightarrow \Delta s = \frac{2\pi}{k} \cdot 10^9 = 1.710 \times 10^9 \text{ cm}$$

THE EQUILIBRIUM TEMPERATURE IS

$$T_{\text{EQ}} = X_0 \cdot 10^6 = \left(\frac{3}{10}\right)^{2/5} \times 10^6 = 6.18 \times 10^5 \text{ K}$$



LOOK AT loop Phase Space.m

```

% Map out the phase space of a loop with a given pressure and heating rate.
% CCK 2009-Apr-02 Based on looptest2.m
% function loopPhaseSpace

% ODE solver options
lsode_options('integration method','adams')% default solver is
'bdf'='stiff'.
lsode_options('maximum order',-1)           % -1 means unconstrained.
lsode_options                               % display all the settings.

% Set the global parameters
global T0
global ne0
global EH
global kappa
P0 = 1e16 % Pressure expressed as ne*T
T0 = 1e6
ne0 = P0/T0
EH = 1e-3
kappa = 1e-6; % Plasma thermal conductivity (normally 1e-6)

% Position array
Npoints = 100
s1 = 1e5 % Minimum value of coordinate axis (1e5 cm = 1 km)
sN = 1e9 % 1.2e9 % Maximum value of coordinate axis (1e8 cm = 1 Mm)
s = exp(linspace(log(s1), log(sN), Npoints));

% Integrate from loop top to footpoint
FootpointTemperatures = ((1:6) - 1/2)*1e5 ;
colors = ['k-';'r-';'g-';'c-';'b-';'m-'];
gradTfoot = 0.00
hold off
for i=1:6
    Tfoot = FootpointTemperatures(i)
    q0 = [Tfoot, gradTfoot];
    q = lsode(@loop, q0, s);
    T = q(:, 1);
    gradT = q(:, 2);
    plot(T/1e6, gradT*1e8/1e6, colors(i,:))
    hold on
    plot(T/1e6, -gradT*1e8/1e6, colors(i,:))
end
xlabel('T (MK)', 'fontsize', 18)
ylabel('dT/ds (MK/Mm)', 'fontsize', 18)
title('Loop Models: E_H = 10^{-3}, n_eT = 10^{16}', 'fontsize', 18)
axis([0,1,-1,1])

```


LOOP THERMAL STABILITY

NOW, SUPPOSE THAT WE TAKE A TUBE OF PLASMA AT $T = T_{EQ}$ AND IMPOSE A SINUSOIDAL PERTURBATION WITH A PERIOD LESS THAN ΔS . WHAT HAPPENS?

- RADIATIVE TERM IS THE SAME AS FOR AN EXACT SOLUTION TO THE ENERGY BALANCE EQUATION.
- TEMPERATURE GRADIENTS ARE STEEPER, SO F_c AND $\nabla \cdot \vec{F}_c$ ARE INCREASED.
- ∴ CONDUCTION WILL SMOOTH OUT THE GRADIENT.

IF THE PERTURBATION HAS A PERIOD LONGER THAN ΔS ,

- F_c AND $\nabla \cdot \vec{F}_c$ DECREASE
- RADIATION REMAINS THE SAME
- ∴ THE AMPLITUDE OF THE PERTURBATION GROWS.

THIS IS THE FAMOUS "THERMAL INSTABILITY" OF THE CORONA.