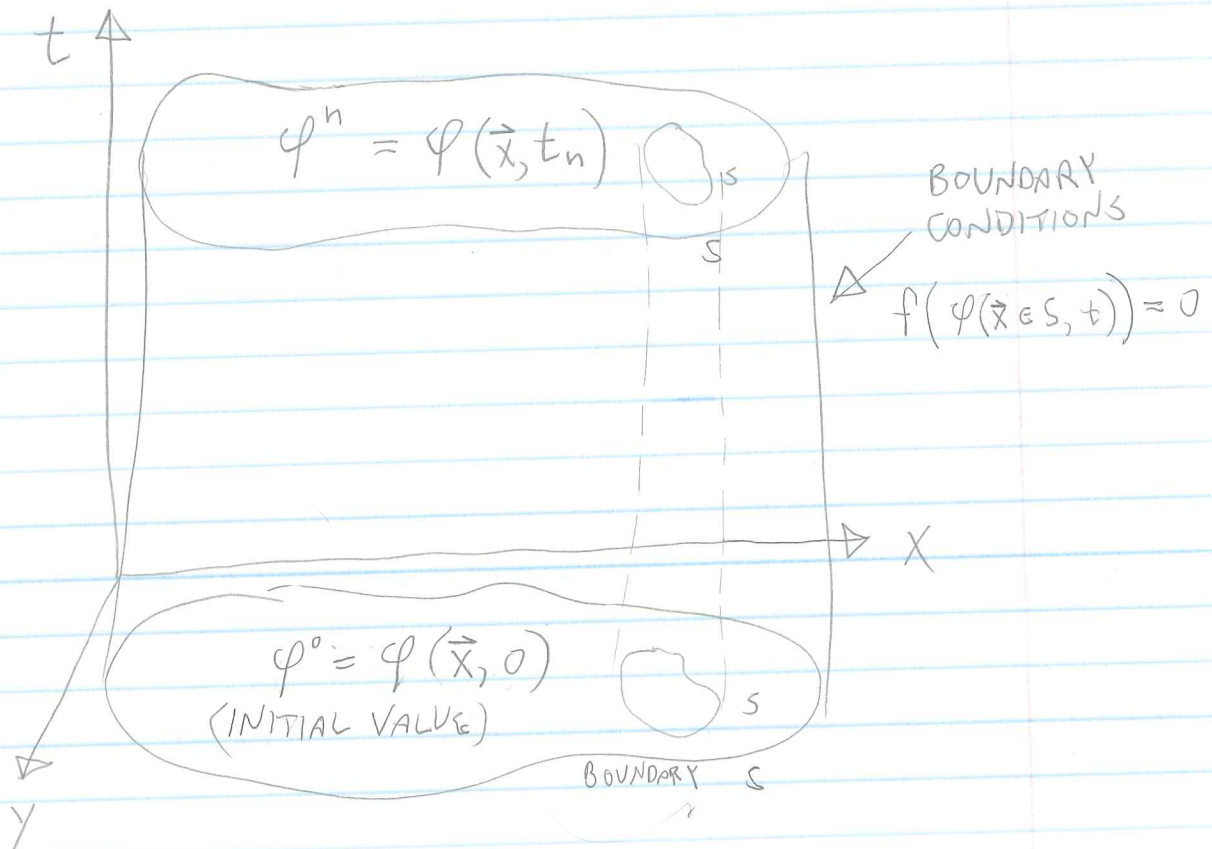


INITIAL VALUE PROBLEMS OUTLINE

- I • FINITE DIFFERENCING
- II • FORWARD TIME - CENTERED SPACE
- DO AN EXAMPLE → BLOW UP
- IV → • LAX METHOD (EXPLICIT DEMO)
- III → • VON NEUMANN STABILITY ANALYSIS
- VI → • NUMERICAL DISSIPATION
- V → • COURANT CONDITION

THE INITIAL VALUE PROBLEM:



CONSIDER A WAVE EQUATION:

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

ANALYTICAL SOLN:

$$\phi = f(x-vt) + g(x+vt)$$

f, g ARBITRARY FUNCTIONS

AS WITH O.D.E., WANT TO MAKE IT 1ST ORDER IN TIME:

LET $\psi = \frac{\partial \phi}{\partial t}$. THEN:

$$\frac{\partial \psi}{\partial t} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\partial \phi}{\partial t} = \psi$$

AND WE MAY ATTACK THIS IN THE USUAL MANNER OF AN ODE INITIAL VALUE PROBLEM.

BUT HOW DO WE EVALUATE THE R.H.S.?

★ HOW TO EVALUATE $\frac{\partial^2 \phi}{\partial x^2}$??

FINITE-DIFF:

$$\phi(x) \rightarrow \phi_i, \quad x_i = i\Delta$$

TRY $\frac{\partial^2 \phi_i}{\partial x^2} \approx \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta^2}$ (HOW DID I COME UP WITH THAT?)

TAYLOR SERIES:

$$\phi_{i+1} = \phi_i + \Delta \phi_i' + \frac{1}{2} \Delta^2 \phi_i'' + \frac{1}{6} \Delta^3 \phi_i''' + \frac{1}{24} \Delta^4 \phi_i^{(4)}$$

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta^2} = \phi_i'' + \frac{1}{12} \Delta^2 \phi_i^{(4)} + \dots$$

TRUNCATION ERROR

(NOTE HOW ODD POWERS OF Δ CANCEL!)

THIS SCHEME IS CALLED "CENTERED SPACE"

TIME EVOLUTION:

TRY **EULERIAN**:

$$\psi_j(t+\delta) = \psi_j(t) + \delta \dot{\psi}_j(t) + \frac{1}{2} \delta^2 \ddot{\psi}_j(t) + \dots$$

INDEX REFERS TO POSITION TRUNCATION ERROR

(AND SIMILARLY FOR $\phi_j(t+\delta)$)

OR SIMPLY:

$$\psi_j^{n+1} = \psi_j^n + \delta \dot{\psi}_j^n$$

$$\psi_j^{n+1} = \psi_j^n + \delta \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta^2} v^2 \quad \text{(FOR OUR WAVE EQN.)}$$

$$\phi_j^{n+1} = \phi_j^n + \delta \dot{\phi}_j^n$$

$$\phi_j^{n+1} = \phi_j^n + \delta \psi_j^n$$

THESE ARE JUST A BIG MATRIX EQN.,

$$\begin{pmatrix} \phi^{n+1} \\ \psi^{n+1} \end{pmatrix} = A \begin{pmatrix} \phi^n \\ \psi^n \end{pmatrix}$$

PROPAGATOR (CF. ~~matrix~~ $e^{-iH\delta/\hbar}$)

~~$\begin{pmatrix} \phi^m \\ \psi^m \end{pmatrix} = A \begin{pmatrix} \phi^n \\ \psi^n \end{pmatrix}$~~

THE BEST WAY TO GET INSIGHT ABOUT THE PROPAGATOR A IS TO FIND ITS EIGENVALUES AND EIGENVECTORS:

$$\text{IF } A \vec{u} = \xi \vec{u},$$

THEN IF WE CHOOSE AN INITIAL CONDITION $\begin{pmatrix} \phi^0 \\ \psi^0 \end{pmatrix} = \vec{u}$,

ONE TIMESTEP WILL JUST MULTIPLY THE STATE BY THE EIGENVALUE:

$$\begin{pmatrix} \phi^1 \\ \psi^1 \end{pmatrix} = \xi \begin{pmatrix} \phi^0 \\ \psi^0 \end{pmatrix}$$

LIKEWISE, THE STATE AT ANY FUTURE TIME WOULD BE JUST:

$$\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix} = \xi^{t/\Delta t} \begin{pmatrix} \phi^0 \\ \psi^0 \end{pmatrix}$$

AND, OF COURSE, EIGENVECTORS MAKE UP A BAZIS. SO THIS TIME BEHAVIOR CAN BE USED TO CHARACTERISE THE RESPONSE OF YOUR DIFFERENCING SCHEME TO ALL POSSIBLE INITIAL CONDITIONS.

TAKING OUT THE DIMENSIONS ...

~~WITHOUT LOSS OF GENERALITY, LET~~
 $\Delta = 1$
~~(FREE CHOICE OF UNITS, SINCE THERE ARE NO LENGTH SCALES INHERENT IN THE PDE)~~

NOW CHOOSE "NATURAL" TIME UNITS,

$$1 \equiv \Delta/v$$

I.E. $v = \Delta$. OUR DIFFERENCE EQUATIONS BECOME!

TIMESTEP $\delta \rightarrow \delta v / \Delta$

$$\psi_i^{n+1} = \psi_i^n + \delta (\varphi_{i+1}^n + \varphi_{i-1}^n - 2\varphi_i^n)$$

$$\varphi_i^{n+1} = \varphi_i^n + \delta \psi_i^n$$

~~EXAMPLE~~

~~INITIAL CONDITIONS:~~

~~$\psi^0 = a(1, -1)$~~

~~$\varphi^0 = (1, -1)$~~

~~ASSUME PERIODIC BOUNDARY CONDITIONS!~~

VON NEUMANN STABILITY ANALYSIS

INITIAL CONDITIONS:

$$\psi^0 = B e^{ikj\Delta}, \quad B \in \mathbb{C}$$

$$\varphi^0 = e^{ikj\Delta}$$

$$\lambda = \sqrt{-1}$$

k = WAVENUMBER $\in \mathbb{R}$

j = POSITION, $x = j\Delta$
INDEX

APPLY DIFFERENCE EQUATIONS:

$$\psi^1 = B e^{ikj\Delta} + \delta \left(e^{ik(j+1)\Delta} + e^{ik(j-1)\Delta} - 2 e^{ikj\Delta} \right)$$

$$= B e^{ikj\Delta} + \delta \left(e^{ik\Delta} + e^{-ik\Delta} \right) e^{ikj\Delta} - 2\delta e^{ikj\Delta}$$

$\underbrace{\hspace{10em}}_{2 \cos k\Delta}$

$$= (B + 2\delta \cos k\Delta - 2\delta) e^{ikj\Delta} = \underline{\underline{\left(1 + \frac{2\delta}{B} [\cos k\Delta - 1] \right) \psi^0}}$$

$$\varphi^1 = e^{ikj\Delta} + B\delta e^{ikj\Delta} = \underline{\underline{(1 + B\delta) \varphi^0}}$$

NOW, $\begin{pmatrix} \varphi^0 \\ \psi^0 \end{pmatrix}$ WILL BE AN EIGENVECTOR OF

THE PROPAGATOR A IF:

CANCEL

$$\cancel{1 + \frac{2\delta}{B} [\cos k\Delta - 1]} = \cancel{1 + B\delta} \equiv \xi \quad (\text{EIGENVALUE!})$$

$$-B^2 = 2(\cos k\Delta - 1) \Rightarrow B = \pm i \sqrt{(1 - \cos k\Delta) 2}$$

$$\Rightarrow \xi = 1 \pm i\delta \sqrt{2 - 2\cos k\Delta}$$

~~FOR THE POSITIVE ROOT, $\xi = 1 + i\delta \sqrt{2 - 2\cos k\Delta} \Rightarrow$~~ FTCS
 $|\xi| > 1$ FOR MOST VALUES OF k AND δ .

$$\begin{pmatrix} \varphi^1 \\ \psi^1 \end{pmatrix} = A \begin{pmatrix} \varphi^0 \\ \psi^0 \end{pmatrix} = \xi \begin{pmatrix} \varphi^0 \\ \psi^0 \end{pmatrix}$$

SIMILARLY,

$$\begin{pmatrix} \varphi^n \\ \psi^n \end{pmatrix} = \xi^n \begin{pmatrix} \varphi^0 \\ \psi^0 \end{pmatrix}$$

THUS, IF $|\xi| > 1$, A IS "UNSTABLE"
FOR THE "EIGENMODE" (φ_0, ψ_0) .

OUR "FORWARD TIME / CENTERED SPACE"
DIFFERENCING SCHEME IS UNFORTUNATELY UNSTABLE
FOR ALMOST ALL h .

NOTE THAT THE RANGE OF h IS GIVEN BY

$$0 < h\Delta < \pi$$

$$0 < h < \frac{\pi}{\Delta} = \text{NYQUIST FREQUENCY.}$$

ξ IS MAXIMUM AT THE NYQUIST FREQUENCY.

THIS IS TYPICAL.

WHAT DO \pm ROOTS FOR B CORRESPOND TO?
→ DIFFERENT PHASES OF ψ & φ . SOME MODES
ARE STABLE, OTHERS DECAY.