

LAST TIME:

INVESTIGATED SCALAR WAVE EQN.

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

OR

$$\frac{\partial \psi}{\partial t} = v^2 \frac{\partial \phi}{\partial x^2}, \quad \frac{\partial \phi}{\partial t} = \psi$$

WE LET $t = n\delta$, $x = j\Delta$. FTCS SCHEME
IN UNITS WHERE $v = \Delta$:

$$\psi_j^{n+1} = \psi_j^n + \delta (\phi_{j+1}^n + \phi_{j-1}^n - 2\phi_j^n)$$

$$\phi_j^{n+1} = \phi_j^n + \delta \psi_j^n$$

THEN USING VON NEUMANN'S TRICK, WE FOUND THAT
THIS SCHEME IS UNSTABLE!

USING $\psi^0 = B e^{ikj\Delta}$, $\phi^0 = e^{ikj\Delta}$, $B \in \mathbb{C}$,

$$\begin{bmatrix} \psi^n \\ \phi^n \end{bmatrix} = \xi^n \begin{bmatrix} \psi^0 \\ \phi^0 \end{bmatrix}$$

WHERE $\xi = 1 \pm i\delta \sqrt{2-2\cos k\Delta}$

$$|\xi| = |1 + \delta^2(2-2\cos k\Delta)| \geq 1 \Rightarrow \text{UNSTABLE.}$$

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WHAT ABOUT THIS INSTABILITY PROBLEM?

QUICK FIX DUE TO MR. LAX:

FIRST, WE'LL CHANGE OUR SPATIAL DIFFERENCING BY REARRANGING THE EQUATIONS:

WE HAD $\ddot{\phi} = v^2 \phi''$ (1D. SCALAR WAVE EQN.)

LET $r \equiv \dot{\phi}$, $s \equiv v \phi'$ PREVIOUSLY ψ

$$\dot{r} = v s'$$

$$\dot{s} (= v \dot{\phi}') = v r'$$

$$\frac{\partial}{\partial t} \begin{pmatrix} r \\ s \end{pmatrix} = v \begin{pmatrix} s' \\ r' \end{pmatrix}$$

IT IS EASY TO SHOW THAT FTCS IS (ONCE AGAIN) UNSTABLE FOR THIS SET OF EQUATIONS (NR § 19.1). FTCS:

$$r_j^{n+1} = r_j^n + \delta \dot{r}_j^n = r_j^n + \delta v \frac{s_{j+1}^n - s_{j-1}^n}{2\Delta}$$

$$s_j^{n+1} = s_j^n + \delta \dot{s}_j^n = s_j^n + \delta v \frac{r_{j+1}^n - r_{j-1}^n}{2\Delta}$$

$$\text{LET } r_j^0 = B e^{ikj\Delta}, \quad s_j^0 = e^{ik\Delta j}$$

$$r_j^1 = B e^{ikj\Delta} + \frac{\delta v}{2\Delta} \left(e^{ik\Delta(j+1)} - e^{ik\Delta(j-1)} \right)$$

$$= r_j^0 + \frac{\delta v}{2\Delta B} \underbrace{\left(e^{ik\Delta} - e^{-ik\Delta} \right)}_{2i \sin k\Delta} r_j^0$$

$$\text{EIGENVALUE } \xi = 1 + \frac{\delta v}{\Delta B} i \sin k\Delta \quad *$$

AND NOW,

$$s_j^1 = e^{ik\Delta j} + \frac{\delta v}{2\Delta} B \left(e^{ik(j+1)\Delta} - e^{ik(j-1)\Delta} \right)$$

$$= s_j^0 + \frac{\delta v}{2\Delta} B (2i \sin k\Delta) s_j^0$$

$$\xi = 1 + \frac{\delta v}{\Delta} B i \sin k\Delta \quad **$$

SETTING $* = **$, $B = \pm 1$.

$$\xi = 1 \pm i \frac{\delta v}{\Delta} \sin k\Delta$$

$$|\xi|^2 = 1 + \frac{\delta^2 v^2}{\Delta^2} \sin^2 k\Delta > 1 \quad \text{FOR MOST } k.$$

UNSTABLE.

THE LAX METHOD STABILIZES THESE DIFFERENCE EQUATIONS BY THE SUBSTITUTIONS:

$$r_j^n \rightarrow \frac{1}{2} (r_{j+1}^n + r_{j-1}^n)$$

$$s_j^n \rightarrow \frac{1}{2} (s_{j+1}^n + s_{j-1}^n)$$

SO

$$r_j^{n+1} = \frac{1}{2} (r_{j+1}^n + r_{j-1}^n) + \frac{1}{2} \delta (s_{j+1}^n - s_{j-1}^n)$$
$$s_j^{n+1} = \frac{1}{2} (s_{j+1}^n + s_{j-1}^n) + \frac{1}{2} \delta (r_{j+1}^n - r_{j-1}^n)$$

WHERE WE'VE AGAIN USED TIME UNITS $\frac{\Delta}{v} \rightarrow 1$.

VON NEUMANN:

LET $r_j^0 = B e^{ikj\Delta}$, $s_j^0 = e^{ikj\Delta}$

$$r_j^1 = \frac{B}{2} (e^{ik(j+1)\Delta} + e^{ik(j-1)\Delta}) + \frac{1}{2} \delta (e^{ik(j+1)\Delta} - e^{ik(j-1)\Delta})$$

$$= (B \cos k\Delta - i\delta \sin k\Delta) e^{ikj\Delta}$$

$$= (\cos k\Delta - i \frac{\delta}{B} \sin k\Delta) r_j^0$$

$$s_j^1 = (\cos k\Delta - i\delta B \sin k\Delta) s_j^0$$

EIGENVALUES:

$$\xi = \cos k\Delta - i \frac{\delta}{B} \sin k\Delta = \cos k\Delta - i \delta B \sin k\Delta$$

$$\Rightarrow \underline{B = \pm 1.}$$

$$\xi = \cos k\Delta \mp i \delta \sin k\Delta$$

STABILITY REQUIREMENT:

$$|\xi| \leq 1$$

$$\cos^2 k\Delta + \delta^2 \sin^2 k\Delta \leq 1$$

$$\Rightarrow \delta < 1 \quad (\text{ASSUME } \delta > 0)$$

RECALL THAT OUR TIME UNITS ARE $\frac{\Delta}{v}$, SO

WHAT WE REALLY HAVE IS:

$$\boxed{\delta \leq \frac{\Delta}{v}}, \quad \frac{\Delta}{\delta} \geq v$$

**COURANT
CONDITION**

$$\text{LET } \delta = \frac{1}{S} \frac{\Delta}{v}$$

$S > 1$ "SAFETY FACTOR"

SPEED OF INFORMATION
PROPAGATION ON GRID. IF
LESS THAN WAVE SPEED v ,
THEN POWER PILES UP!

Δ/v IS
A HELPFUL HEURISTIC
FOR VARIABLE
STEPSIZE WHEN
 v IS THE FASTEST
PHYSICAL PROPAGATION
SPEED IN THE
SYSTEM.

ANSWERS TO QUESTIONS THAT CAME UP SINCE MONDAY

① $|\xi| \leq 1$ IS GENERALLY CONSIDERED "GOOD", BUT OF COURSE YOU'D BE DISAPPOINTED IF $\xi = 0.1$!!!

② HOW DO I KNOW WHETHER TO TRUST* THE RESULTS OBTAINED NUMERICALLY?

- [1] COMPARE TO KNOWN PHYSICS (E.G. LIMITING CASES)

- [2] SIMPLIFIED NUMERICAL EXPERIMENTS "TEST CASES"

[3] INVARIANTS CONSERVED?

[4] IS MY INSTABILITY PHYSICS OR NUMERICS?

→ ANALYTICAL STABILITY CALCULATIONS

→ MICROSCOPIC BEHAVIOR OF NUMERICS, OR VON NEUMANN ANALYSIS

→ WHAT WAVENUMBERS ARE AMPLIFIED?

* ③ CONTRAST FIDELITY TO ACCURACY. WOULD YOU RATHER HAVE $\text{EPS} = 10^{-7}$ OR CONSERVE MOMENTUM?

E.G. SMOOTH PARTICLE HYDRO!

LAX METHOD — NUMERICAL DISSIPATION
EQUIVALENT TO FTCS + DIFFUSION TERM!

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LOOK AGAIN AT DIFFERENCE EQUATION:

$$\begin{aligned} r_j^{n+1} &= \frac{1}{2} (r_{j+1}^n + r_{j-1}^n) + \frac{1}{2} \delta (s_{j+1}^n - s_{j-1}^n) \\ &= \underbrace{r_j^n + \frac{1}{2} \delta (s_{j+1}^n - s_{j-1}^n)}_{\text{FTCS}} + \underbrace{\frac{1}{2} (r_{j+1}^n - 2r_j^n + r_{j-1}^n)}_{\text{REMAINDER TERM}} \end{aligned}$$

THIS IS EXACTLY THE FTCS REPRESENTATION OF:

$$\frac{\partial r}{\partial t} = -v \frac{\partial s}{\partial x} + \underbrace{\frac{\Delta^2}{2\delta} \frac{\partial^2 r}{\partial x^2}}_{\text{DIFFUSION TERM}}$$

$$\frac{\Delta^2}{2\delta} = \frac{1}{2} \Delta s v, \quad S \equiv \text{SAFETY FACTOR}, \quad \delta = \frac{1}{S} \frac{\Delta}{v}$$

UBIQUITOUS PROBLEM: (STABLE) DIFFERENCING SCHEMES
LEAD TO DISSIPATION TERMS

- SCALAR WAVE EQN : NUMERICAL DIFFUSION
- FLUID MECH : NUMERICAL VISCOSITY
- MHD : NUMERICAL RESISTIVITY

WARNING! THESE WORDS "DIFFUSION", "VISCOSITY", ETC.
COME FROM AN ANALOGY (IN THIS CASE BASED ON FTCS!).
THE "DIFFUSION COEFFICIENT" ISN'T AN ACCURATE
ASSESSMENT OF THE BEHAVIOR OF THE SCHEME.
RELY ON EIGENVALUES TO STUDY SCHEME BEHAVIOR!

AMPLITUDE AND PHASE ERROR

(A CLOSER LOOK AT VON NEUMANN ANALYSIS)

WE HAVE BEEN LOOKING AT A SCALAR WAVE EQUATION,

$$\frac{\partial^2 \phi}{\partial t^2} = v^2 \frac{\partial^2 \phi}{\partial x^2}$$

IT PROVED ADVANTAGEOUS TO DEFINE NEW VARIABLES,

$$r = \dot{\phi}, \quad s = v \phi'$$

$$\rightarrow \boxed{\frac{\partial}{\partial t} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} s' \\ r' \end{pmatrix}}$$

USING THE LAX INTEGRATION SCHEME, WITH $x = j\Delta$ AND $t = n\delta$,

$$\boxed{\begin{aligned} r_j^{n+1} &= \frac{1}{2} (r_{j+1}^n + r_{j-1}^n) + \frac{1}{2} \delta (s_{j+1}^n - s_{j-1}^n), \\ s_j^{n+1} &= \frac{1}{2} (s_{j+1}^n + s_{j-1}^n) + \frac{1}{2} \delta (r_{j+1}^n - r_{j-1}^n), \end{aligned}}$$

AND THE ANSATZ

$$r_j^0 = B e^{ikj\Delta}, \quad s_j^0 = e^{ikj\Delta}$$

WE FOUND THAT

$$\begin{pmatrix} r^n \\ s^n \end{pmatrix} = \xi^n \begin{pmatrix} r^0 \\ s^0 \end{pmatrix} \quad \text{IFF} \quad |B| = 1$$

$$\text{AND} \quad \boxed{\xi = \cos k\Delta \mp i\delta \sin k\Delta}$$

$$\text{WHERE } \delta = \frac{\Delta}{v} \delta \quad (\text{TIME UNITS!})$$

WE MAY SOLVE THE ~~THE~~ PDE'S ~~BY A~~ ANALYTICALLY BY A SIMILAR PROCEDURE:

ANSATZ $\begin{cases} s^0 = e^{ikhx} & r^0 = Be^{ikhx} \\ \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r^0 \\ s^0 \end{pmatrix} f(t) \end{cases} \quad \star$

THEN $\dot{r} = ikv e^{ikhx} \cancel{f(t)} = \frac{ikhv}{B} r$

$\dot{s} = iBkhv e^{ikhx} \cancel{f(t)} = iBkhv s$

SO SATISFY \star , IMPLIES

$\Rightarrow B = \pm 1, \quad \cancel{B}$

$\Rightarrow \frac{\partial}{\partial t} \begin{pmatrix} r \\ s \end{pmatrix} = \pm ikv \begin{pmatrix} r \\ s \end{pmatrix}$

$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r^0 \\ s^0 \end{pmatrix} e^{ikh(\pm vt)}$ $\begin{cases} - \text{RIGHT GOING} \\ + \text{LEFT GOING} \end{cases}$

$\begin{pmatrix} r' \\ s' \end{pmatrix} = \left[\cos(\pm kv\delta) + i \sin(\pm kv\delta) \right] \begin{pmatrix} r^0 \\ s^0 \end{pmatrix}$

USING TIME UNITS $\frac{\Delta}{v} = 1, \quad \delta \rightarrow \frac{\Delta}{v} \delta$ (δ IS DIMENSIONLESS)

$\begin{pmatrix} r' \\ s' \end{pmatrix} = \left[\cos(\pm k\Delta\delta) + i \sin(\pm k\Delta\delta) \right] \begin{pmatrix} r^0 \\ s^0 \end{pmatrix}$
 $\rightarrow = \underbrace{\begin{matrix} \text{ANALYTIC} \\ \text{LAX} \end{matrix}}_{\text{LAX}} = \cos(k\Delta\delta) \pm i \sin(k\Delta\delta)$

THIS AGREES WITH $\underbrace{\text{LAX}}_{\text{LAX}}$ FOR $\delta = 1$ (WHY?)

$$\xi = \cancel{|\xi|} |\xi| e^{i\theta}$$

THE $|\xi|$ PART AFFECTS AMPLITUDE
AND STABILITY.

WHAT IS THE \ominus PART?

$$\frac{\xi_{\text{LAX}}}{\xi_{\text{ANALYTIC}}} = \frac{|\xi_{\text{LAX}}|}{\cancel{|\xi_{\text{ANALYTIC}}|}} e^{i(\theta_{\text{LAX}} - \theta_{\text{ANALYTIC}})} \triangleq 1 + \delta$$

PHASE ERROR ACCUMULATES WITH
EACH TIMESTEP!

NOTE THAT $\theta_{\text{LAX}} - \theta_{\text{ANALYTIC}}$ IS

A FUNCTION OF k — SO PHASE
SPEED VARIES WITH k .

"NUMERICAL DISPERSION"

~~EXERCISE: WORK OUT~~ $e^{i(\theta_{\text{LAX}} - \theta_{\text{ANALYTIC}})}$

NOTE THAT, WITH $S = 1$, THERE IS NO NUMERICAL DISPERSION!
IN THAT CASE, THE NUMERICAL SOLN. IS EXACT!