

STABILIZED METHOD FOR

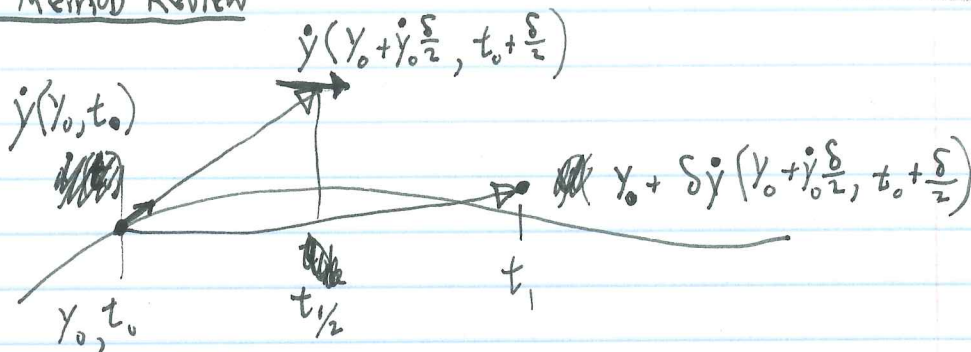
2<sup>ND</sup> ORDER IN TIME: (RK2)

START WITH SPATIAL DIFFERENCE ERS:

$$\dot{\psi}_i = \varphi_{i+1} + \varphi_{i-1} - 2\varphi_i$$

$$\dot{\varphi}_i = \psi_i$$

MIDPOINT METHOD REVIEW



APPLY THIS SCHEME TO OUR PDE:

(1) HALF-STEP USING EULER METHOD

$$\psi_j^{n+k} = \psi_j^n + \frac{\delta}{2} \dot{\psi}_j^n$$

$$\varphi_j^{n+k} = \varphi_j^n + \frac{\delta}{2} \dot{\varphi}_j^n$$

$$\psi_j^n = \varphi_j^n + \frac{\delta}{2} (\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n)$$

$$\dot{\varphi}_j^n = \psi_j^n$$

(2)

EVALUATE DERIVS AT HALF-STEP

$$\dot{\psi}_j^{n+k} = \varphi_{j+1}^{n+k} - 2\varphi_j^{n+k} + \varphi_{j-1}^{n+k}$$

$$= \varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi_j^n + \frac{\delta}{2} (\psi_{j+1}^n + \psi_{j-1}^n - 2\psi_j^n)$$

$$\dot{\varphi}_j^{n+k} = \psi_j^{n+k} = \psi_j^n + \frac{\delta}{2} (\varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi_j^n)$$

(3) ~~TAKE~~ TAKE FULL STEP USING DERIVS FROM (2)

$$\psi_j^{n+1} = \psi_j^n + \delta \dot{\psi}_j^{n+\frac{1}{2}}$$

$$\psi_j^{n+1} = \psi_j^n + \delta \left[ \varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi_j^n + \frac{\delta}{2} (\psi_{j+1}^n + \psi_{j-1}^n - 2\psi_j^n) \right]$$

$$\varphi_j^{n+1} = \varphi_j^n + \delta \dot{\varphi}_j^{n+\frac{1}{2}}$$

$$\varphi_j^{n+1} = \varphi_j^n + \delta \left[ \psi_j^n + \frac{\delta}{2} (\varphi_{j+1}^n + \varphi_{j-1}^n - 2\varphi_j^n) \right]$$

THESE ARE OUR NEW 2<sup>nd</sup> ORDER DIFFERENCE EQUATIONS.

VON NEUMANN SEZ:

INITIAL CONDITIONS:  $\psi^0 = B e^{ikj\Delta}$   
 $\varphi^0 = e^{ikj\Delta}$

APPLY DIFFERENCE EQUATIONS:

$$\psi^1 = B e^{ikj\Delta} + \delta \left[ e^{ik(j+1)\Delta} + e^{ik(j-1)\Delta} - 2 e^{ikj\Delta} \right] + \frac{\delta B}{2} \left( e^{ik(j+1)\Delta} + e^{ik(j-1)\Delta} - 2 e^{ikj\Delta} \right)$$

$$= \left[ B + \delta \left( 1 + \frac{\delta B}{2} \right) (2 \cos k\Delta - 2) \right] e^{ikj\Delta}$$

$$= \left[ 1 + \delta \left( \frac{1}{B} + \frac{\delta}{2} \right) (2 \cos k\Delta - 2) \right] \psi^0$$

$$\varphi^1 = e^{ikj\Delta} + \delta \left[ B e^{ikj\Delta} + \frac{\delta}{2} (2 \cos k\Delta - 2) e^{ikj\Delta} \right]$$

$$= \left[ 1 + \delta \left[ B + \frac{\delta}{2} (2 \cos k\Delta - 2) \right] \right] \varphi^0$$

AND AS USUAL, WE SOLVE FOR B BY REQUIRING  $\begin{pmatrix} \psi^0 \\ \varphi^0 \end{pmatrix}$  BE AN EIGENVECTOR OF THE DIFFERENCE EQUATION:

$$\left( \frac{1}{B} + \frac{\delta}{2} \right) (2 \cos k\Delta - 2) = B + \frac{\delta}{2} (2 \cos k\Delta - 2)$$

$$\cancel{B} + \frac{\delta}{2} = B + \frac{\delta}{2} (2 \cos k\Delta - 2)$$

$$B^2 = 2 \cos k\Delta - 2$$

$$B = \pm \sqrt{2 \cos k\Delta - 2}$$



AND THE EIGENVALUES ARE

$$\xi = 1 + \delta \left( B + \frac{\delta}{2} (2 \cos k\Delta - 2) \right)$$

$$= 1 + \delta \frac{\delta^2}{2} (2 \cos k\Delta - 2) \pm \delta \sqrt{2 \cos k\Delta - 2}$$

$$= 1 + \frac{\delta^2}{2} (2 \cos k\Delta - 2) \pm i \delta \sqrt{2 - 2 \cos k\Delta}$$

$$|\xi|^2 = \left[ 1 - \delta^2 (1 - \cos k\Delta) \right]^2 + \delta^2 (2 - 2 \cos k\Delta)$$

$$|\xi|^2 = 1 + \delta^4 (1 - \cos k\Delta)^2 > 1$$

→ SO WHAT?! THIS LOOKS UNSTABLE!

→ BUT NOT VERY :  $\delta^4$ .

IN THE DERIVATION OF THE DIFFERENCE EQUATIONS USING RK2, I CAREFULLY SEGREGATED  $\delta$  FROM  $\delta/2$  ... FOR A GOOD REASON! ⑤

### LONGCOPE DIFFERENCING

BACK UP TO SOLVING FOR B. USE ~~1/2~~ A HALF-STEP THAT IS A BIT LONG:

$$\frac{\delta}{2} \rightarrow (1+\epsilon) \frac{\delta}{2}$$

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$$\left( \frac{1}{B} + (1+\epsilon) \frac{\delta}{2} \right) (2 \cos k\Delta - 2) = B + (1+\epsilon) \frac{\delta}{2} (2 \cos k\Delta - 2)$$

$$\cancel{B} B = \pm \sqrt{2 \cos k\Delta - 2} = \pm i \sqrt{2 - 2 \cos k\Delta}$$

(UN-CHANGED!)

EIGENVALUES:

$$\xi = 1 + \delta \left[ B + (1+\epsilon) \frac{\delta}{2} (2 \cos k\Delta - 2) \right]$$

$$\cancel{= 1 + \delta - \delta^2}$$

$$= 1 - (1+\epsilon) \delta^2 (1 - \cos k\Delta) \pm i \delta \sqrt{2 - 2 \cos k\Delta}$$

$$|\xi|^2 = \left[ 1 - (1+\epsilon) \delta^2 (1 - \cos k\Delta) \right]^2 + \delta^2 (2 - 2 \cos k\Delta)$$

$$= 1 - \underbrace{2\epsilon \delta^2 (1 - \cos k\Delta)}_A + \underbrace{(1+\epsilon)^2 \delta^4 (1 - \cos k\Delta)^2}_B$$

$$|\xi| \leq 1 \quad \underline{\text{IF:}} \quad \text{TERM A} > \text{TERM B}$$

i.e. ~~2\epsilon \geq (1+\epsilon)^2 \delta^2 (1 - \cos k\Delta)~~  $2\epsilon \geq (1+\epsilon)^2 \delta^2 (1 - \cos k\Delta)$  2 NOT MOST!

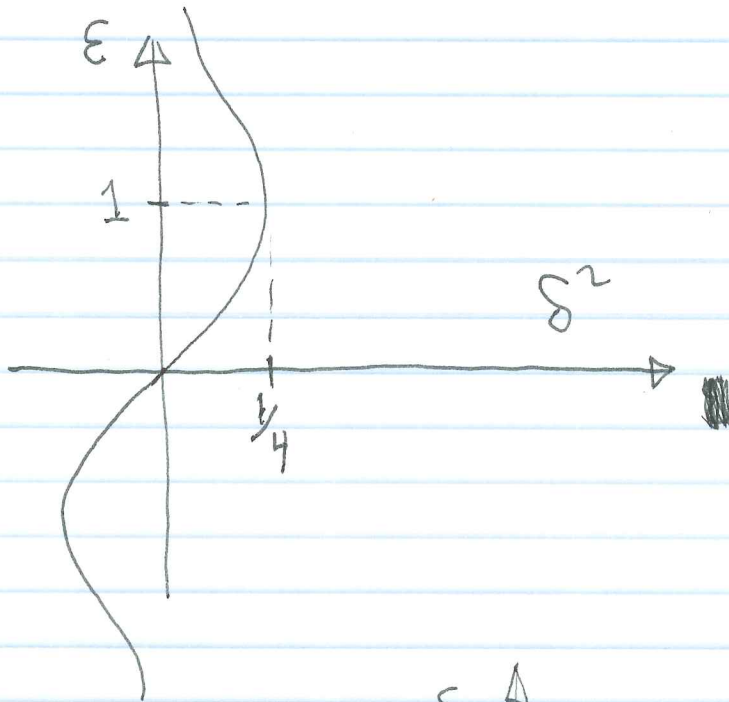
$$\epsilon \geq (1+\epsilon)^2 \delta^2$$

$$\Rightarrow \boxed{\epsilon > 0}$$

AND ...

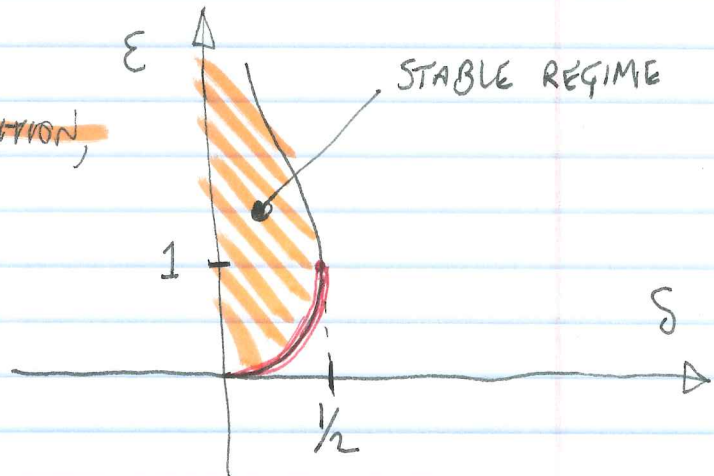
$$\delta^2 \leq \frac{\epsilon}{(1+\epsilon)^2}$$

$$\delta \leq \frac{\sqrt{\epsilon}}{1+\epsilon} \leq \frac{1}{2}$$



COMPARE COURANT CONDITION,

$$\delta \leq 1$$



ASIDE:  $\delta = 1/2, \epsilon = 1$  RESEMBLES  
IMPLICIT EULER (SORT OF)

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ANALYTIC SOLN FOR  $\varepsilon$ :

$$\varepsilon \approx \frac{1 - 2\delta^2 - \sqrt{1 - 4\delta^2}}{2\delta^2}$$

SELECTS LOWER BRANCH

THIS EXPRESSION  
IS FAR FROM  
TRANSPARENT!

$$\text{BUT } \sqrt{1 - 4\delta^2} \approx 1 - 2\delta^2 - 2\delta^4 - 4\delta^6 - 10\delta^8 - 28\delta^{10}$$

$$\varepsilon \approx \delta^2 + 2\delta^4 + 5\delta^6 + 14\delta^8 + \dots$$

RULE OF THUMB:

$$\varepsilon \sim 2\delta^2, \quad \delta < 0.4$$

$$\text{I LIKE } \delta = \frac{1}{5}, \quad \varepsilon = \frac{1}{10}.$$



NOTE THAT THE EIGENVALUE  $\xi$  DEPENDS ON  $k$  !!!

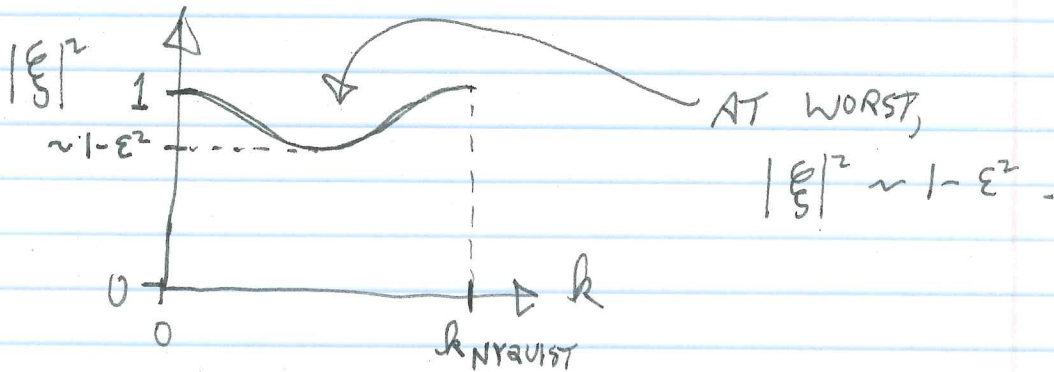
IF WE TAKE  $\delta^2 = \epsilon / (1 + \epsilon)^2$ ,

$$|\xi|^2 = 1 - 2 \frac{\epsilon^2}{(1 + \epsilon)^2} (1 - \cos k\Delta) + \frac{\epsilon^2}{(1 + \epsilon)^2} (1 - \cos k\Delta)^2$$

$$= 1 + \frac{\epsilon^2}{(1 + \epsilon)^2} [1 - 2 \cos k\Delta + \cos^2 k\Delta - 2 + 2 \cos k\Delta]$$

$$|\xi|^2 = 1 - \frac{\epsilon^2}{(1 + \epsilon)^2} \sin^2 k\Delta$$

PHYSICAL INTERPRETATION: WAVE ENERGY DECREASES BY FACTOR  $|\xi|^2$  WITH EACH TIMESTEP.



AFTER  $N_D$  TIMESTEPS, WAVE ENERGY DROPS BY  $1/e$ :

$$N_D = \frac{-1}{\ln(|\xi|^2)} \approx \frac{-1}{\ln(1 - \epsilon^2)} \approx \frac{1}{\epsilon^2} \sim \frac{1}{\delta^4}$$

GIVEN  $\epsilon \sim \delta^2$ , AND  $\delta = \frac{\Delta t v}{\Delta x}$ ,

$$t_D = N_D \Delta t \sim \frac{\Delta x^4}{\Delta t^3 v^4} \sim \frac{1}{\delta^3} \text{ IN NATURAL UNITS.}$$

DAMPING TIME ~~(VERY ROUGH!)~~ (VERY ROUGH!)



EXAMPLE:

SAFETY FACTOR  $S \equiv \frac{1}{\delta} = 4.$

$$n_D \sim \frac{1}{\delta^4} \sim 256$$

$$t_D \sim 64 \frac{\Delta x}{v}$$

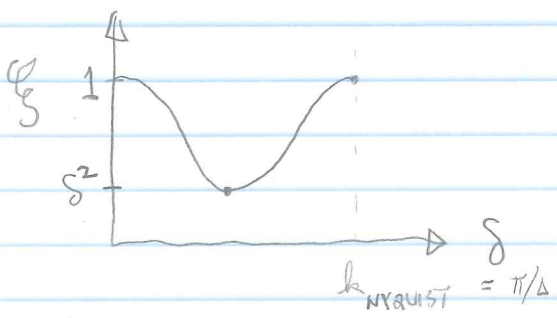
→ WAVE DAMPS TO  $\frac{1}{e}$  ENERGY AFTER  
CROSSING 64 GRID POINTS !!!

REMEMBER THE LAX SCHEME? FOR OUR SAME SCALAR WAVE EQUATION, LAX DIFFERENCING HAS AN EIGENVALUE  $\xi$  GIVEN BY:

$$\xi = \cos k\Delta - i\delta \sin k\Delta$$

$$|\xi|^2 = \cos^2 k\Delta + \delta^2 \sin^2 k\Delta$$

$$= 1 - (1-\delta^2) \sin^2 k\Delta \geq \delta^2$$



THIS LOOKS LIKE HEAVY DAMPING, UNLESS WE USE  $\delta \sim 0.9$  ( $s \sim 1.1$ ).

$$n_D = \frac{-1}{\ln(|\xi|^2)} = \frac{-1}{2 \ln \delta} = \frac{1}{2 \ln s}$$

$$t_D = \frac{\delta}{2 \ln s} = \frac{\Delta}{2v s \ln s}$$

THESE ARE EVIL --- NUMERICAL DISSIPATION IN THE LAX SCHEME PUSHES US TOWARD LOW SAFETY FACTORS. THE LONGCOPE METHOD IS MUCH BETTER FROM THIS POINT OF VIEW!

NOTE: THE FOLLOWING USES NOTATION FROM THE JAN-30 LECTURE ON RK2

(11)

IS OUR STABILIZED "RK2" STILL 2<sup>nd</sup> ORDER? YES!

SUPPOSE WE HAVE AN ODE,

$$\frac{d}{dx} Y(x) = f(Y, x)$$

← COULD HAVE  
 $\vec{y} = \begin{bmatrix} \phi \\ \psi \end{bmatrix}$ ,  
AS IN OUR PDE.

TAYLOR EXPANSION

$$Y_1 = Y_0 + Y_0' \delta + \frac{1}{2} Y_0'' \delta^2 + \mathcal{O}(\delta^3)$$

$$Y_0' = f(Y_0, X_0)$$

$$Y_0'' = \left. \frac{df}{dx} \right|_{x_0, y_0} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = (f_x + f f_y)_{x_0, y_0}$$

THUS,

$$Y_1 = Y_0 + \delta f + \frac{\delta^2}{2} (f_x + f f_y)_{x=0} + \mathcal{O}(\delta^3)$$

WE PROPOSE TO USE A STABILIZED MIDPOINT METHOD:

$$Y_1 \approx Y_0 + \delta f(Y_0 + \alpha \delta f_0, t_0 + \alpha \delta), \quad \alpha \equiv \frac{1+\epsilon}{2}$$

TAYLOR EXP:

$$\begin{aligned} Y_1 &\approx Y_0 + \delta [f + \alpha \delta f_x + \alpha \delta f f_y + \mathcal{O}(\delta^2)] \\ &\approx Y_0 + \delta f + \alpha \delta^2 (f_x + f f_y) + \mathcal{O}(\delta^3). \end{aligned}$$

WE HAVE  $\alpha = \frac{1+\epsilon}{2}$ . LET  $\epsilon = 2\delta^2$  (RULE OF THUMB).

ALL OF THE  $\epsilon$  CONTRIBUTIONS BECOME  $\mathcal{O}(\delta^4)$ . SO EVEN THE THIRD ORDER TERM IS NOT AFFECTED. QED.