

DIFFUSION

$$\frac{\partial u}{\partial t} = \nabla \cdot \vec{F}_u = \nabla \cdot \left(\underset{\substack{\uparrow \\ \text{muss BE } K(u,t)}}{K} \nabla u \right)$$

1D:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial u}{\partial x} \right) = K \frac{\partial^2 u}{\partial x^2} \quad (K = \text{const.})$$

FTCS:

$$u_j^{n+1} = u_j^n + \delta K \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta^2}$$

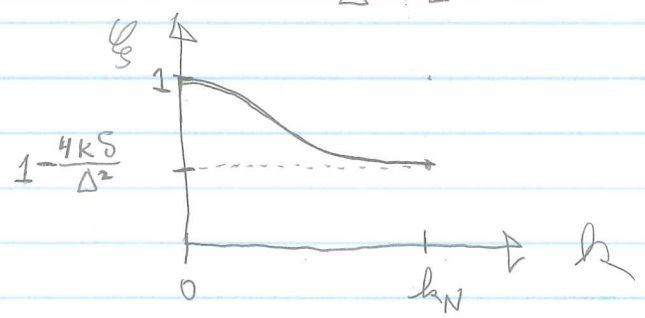
VON NEUMANN:

$$u_j^0 = e^{-ikj\Delta}$$

$$u_j^1 = e^{-ikj\Delta} + \frac{\delta K}{\Delta^2} \left[e^{-ik(j+1)\Delta} + e^{-ik(j-1)\Delta} - 2e^{-ikj\Delta} \right]$$

$$= u_j^0 \left\{ 1 + \frac{\delta K}{\Delta^2} \left[2\cos k\Delta - 2 \right] \right\}$$

$$\xi = 1 - \frac{2K\delta}{\Delta^2} [1 - \cos k\Delta]$$



$$k_N = \frac{\pi}{\Delta}$$

STABILITY:

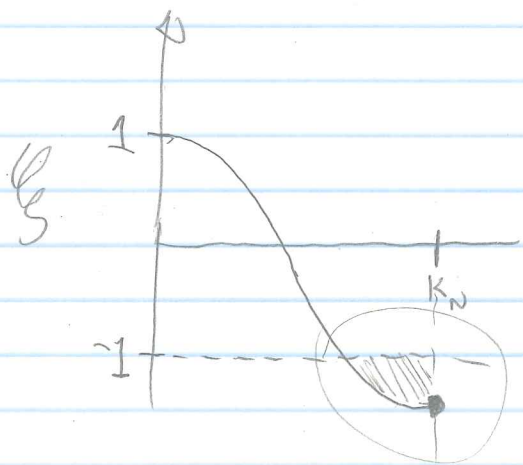
$$|\xi| < 1$$

$$-1 < 1 - \frac{4k\delta}{\Delta^2} < 1$$

↑
ALWAYS TRUE

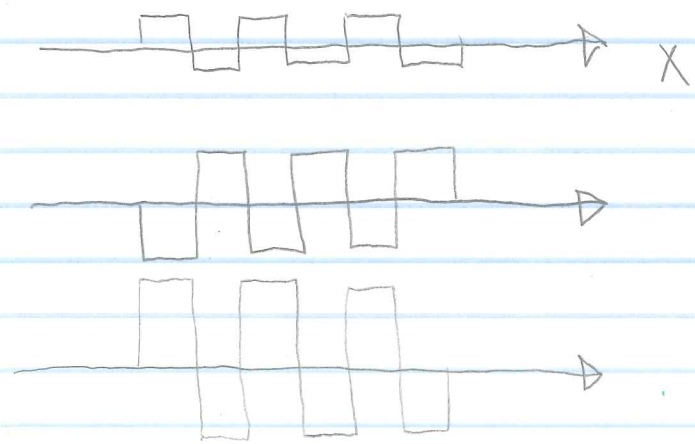
$$\frac{2k\delta}{\Delta^2} < 1$$

INSTABILITY PICTURE



UNSTABLE AT HIGHEST k_N IF $\delta > \Delta^2/2k$.

E.G. AT k_N : OVERSHOTS AND GOES NEGATIVE.



LOOK AT STABILITY CRITERION:

$$\delta < \frac{\Delta^2}{2K} = \text{DIFFUSION TIME FOR } \Delta$$

(RANDOM WALK $X(t) = 2\sqrt{Kt}$)

THIS IS MOST UNFORTUNATE. IF WE ARE INTERESTED IN SPATIAL SCALE l , E.G.

$$l \sim 10 \Delta,$$

THEN $\Delta^2 = .01 l^2$

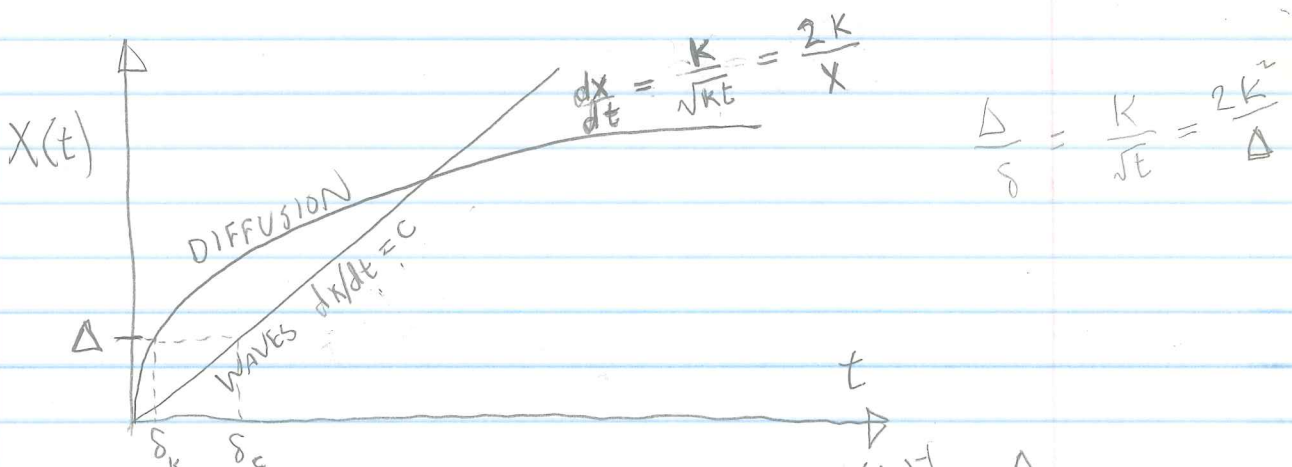
$$\delta < .005 \frac{l^2}{K}$$

STIFF PROBLEM!

THAT IS, WE HAVE A RANGE OF TIMESCALES (CORRESPONDING TO A RANGE OF LENGTH SCALES), AND THE SMALL SCALES DETERMINE THE TIMESTEP EVEN AFTER SMALL SCALE FLUCTUATIONS HAVE DIFFUSED AWAY.

I.E. TINY FRACTION OF DIFFUSION TIME FOR THE PHYSICAL LENGTH SCALE OF INTEREST.

CHARACTERISTIC CURVES FOR PROPAGATION OF INFORMATION:



WAVE EQN "HYPERBOLIC" $\delta < \Delta \left(\frac{dx}{dt} \right)^{-1} = \frac{\Delta}{c}$

DIFFUSION EQN "PARABOLIC" $\delta < \Delta \left(\frac{dx}{dt} \right)^{-1} = \frac{X \Delta}{2K} \Big|_{X=\Delta} = \frac{\Delta^2}{2K}$

"SAFETY FACTOR" $S \equiv \frac{\Delta^2}{2K\delta}$

WE WOULD LIKE A WAY TO TAKE BIGGER
TIMESTEPS!

IMPLICIT FIRST ORDER (ITCS):

$$\textcircled{\star} \quad u_j^{n+1} = u_j^n + \delta k \frac{u_{j+1}^{n+1} + u_{j-1}^{n+1} - 2u_j^{n+1}}{\Delta^2}$$

$$-\frac{\delta k}{\Delta^2} u_{j-1}^{n+1} + \left(1 + \frac{2\delta k}{\Delta^2}\right) u_j^{n+1} - \frac{\delta k}{\Delta^2} u_{j+1}^{n+1} = u_j^n$$

$$A \vec{u}^{n+1} = \vec{u}^n$$

Φ

TRIAGONAL, EASY TO INVERT IN $\mathcal{O}(N_x)$.

$$\vec{u}^{n+1} = A^{-1} \vec{u}^n$$

VON NEUMANN:

$$u_j^0 = e^{ik_j \Delta}$$

$$u_j^1 = \xi u_j^0$$

PLUG INTO $\textcircled{\star}$:

$$\xi e^{ik_j \Delta} = e^{ik_j \Delta} + \frac{\delta k}{\Delta^2} (2 \cos k \Delta - 2) \xi e^{ik_j \Delta}$$

$$\xi \left[1 - \frac{2\delta k}{\Delta^2} (\cos k \Delta - 1) \right] = 1$$

$$\xi = \frac{1}{1 + \frac{2\delta k}{\Delta^2} (1 - \cos k \Delta)}$$

$$\Rightarrow |\xi| < 1$$

UNCONDITIONALLY
STABLE

FIDELITY TO NATURE

{ ~~PHASE ERROR~~ }

$$\frac{\partial \psi}{\partial t} = +K \frac{\partial^2 \psi}{\partial x^2}$$

LET $\psi = A \xi(t) e^{ikx}$

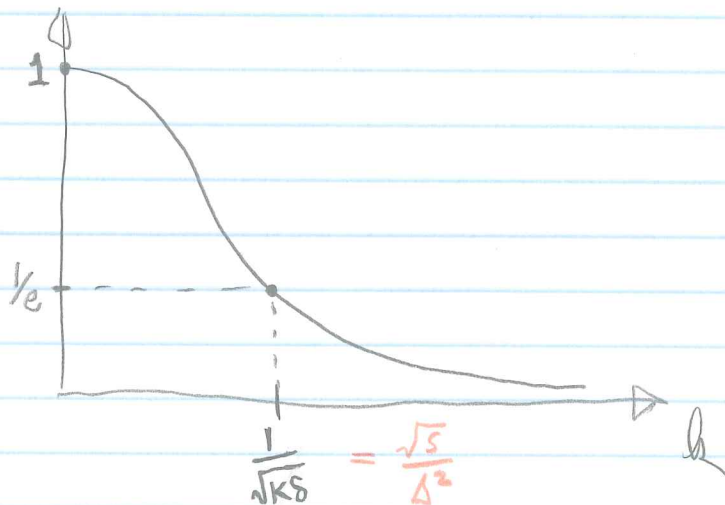
$$\frac{d}{dt} \xi e^{ikx} = -K k^2 \xi e^{ikx}$$

$$\xi = e^{-t/\tau} \quad (\text{LETTING } \xi(0) \equiv 1)$$

$$\tau = \frac{1}{K k^2} = \frac{\lambda^2}{4\pi^2 K}$$

$$\xi(t, k) = e^{-K k^2 t}$$

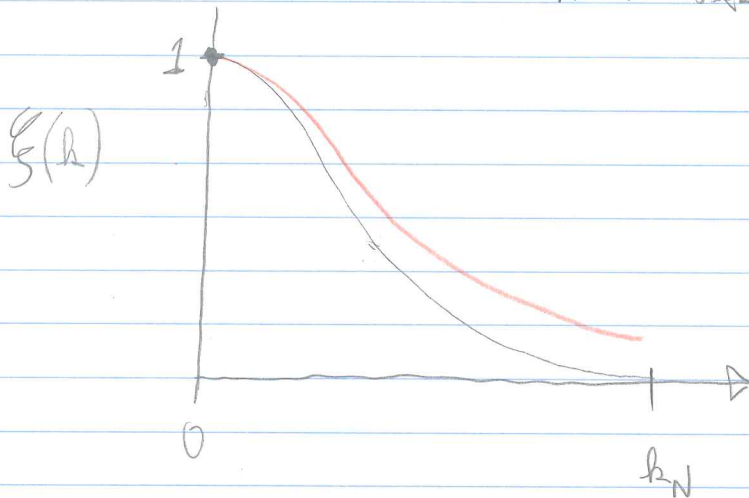
$$\xi(s, k) = e^{-K k^2 s}$$



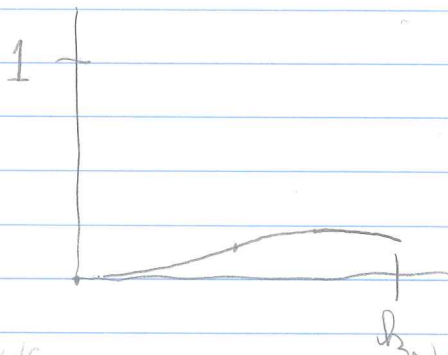
E.g. let $\delta = 10 \frac{\Delta^2}{2k}$ ($\delta = 0.1$)

ANALYTIC SOLN: $\xi = e^{-5k^2\Delta^2}$

IMPLICIT: $\xi = \frac{1}{11 - 10 \cos k\Delta}$



$\xi_{\text{IMPL.}} - \xi_{\text{ANALYT.}}$



IF THE ^{INTERESTING} REAL PHYSICAL SCALE IS $\lambda \approx 10\Delta$, $k \approx \frac{1}{10\Delta}$

THEN THE FRACTIONAL ERROR IN OUR TIMESTEP IS

$$\epsilon = \frac{-\xi_{\text{ANA}} + \xi_{\text{IMP}}}{\xi_{\text{ANA}}} \approx \frac{e^{-5/100} - (11 - 10 \cos \frac{1}{10})^{-1}}{e^{-5/100}} = \frac{0.9512 - 0.9524}{0.9512}$$

$\epsilon \approx 0.0013$