

NOTE: THE GOAL OF THIS EXERCISE IS MAINLY TO GET COMFORTABLE WITH VON NEUMANN ANALYSIS FOR EXPLICIT & IMPLICIT METHODS.

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WE CAN ALSO USE OUR MODIFIED MIDPOINT METHOD TO DO DIFFUSION — $u_j^1 = u_j^0 + \delta \ddot{u}_j^{1/2}$, AS BEFORE...

FIRST $\frac{1}{2}$ STEP:

$$u_j^{1/2} = u_j^0 + (1+\epsilon) \frac{\delta}{2} \overbrace{\frac{k}{\Delta^2} (u_{j+1}^0 + u_{j-1}^0 - 2u_j^0)}^{k \nabla^2 u_j^0} \text{ — DISCUSS.}$$

$$\begin{aligned} \ddot{u}_j^{1/2} &= \frac{k}{\Delta^2} (u_{j+1}^{1/2} + u_{j-1}^{1/2} - 2u_j^{1/2}) \\ &= \frac{k}{\Delta^2} \left[u_{j+1}^0 + (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_{j+2}^0 + u_j^0 - 2u_{j+1}^0) \right. \\ &\quad \left. + u_{j-1}^0 + (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_j^0 + u_{j-2}^0 - 2u_{j-1}^0) \right. \\ &\quad \left. - 2 \left\{ u_j^0 + (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_{j+1}^0 + u_{j-1}^0 - 2u_j^0) \right\} \right] \\ &= \frac{k}{\Delta^2} \left[u_{j+1}^0 + u_{j-1}^0 - 2u_j^0 + (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_{j+2}^0 + u_{j-2}^0 \right. \\ &\quad \left. - 4u_{j+1}^0 - 4u_{j-1}^0 + 6u_j^0) \right] \end{aligned}$$

SO, THE FULL TIMESTEP IS:

$$u_j^1 = u_j^0 + \frac{\delta k}{\Delta^2} \left[u_{j+1}^0 + u_{j-1}^0 - 2u_j^0 + (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_{j+2}^0 + u_{j-2}^0 - 4u_{j+1}^0 - 4u_{j-1}^0 + 6u_j^0) \right]$$

VON NEUMANN ANALYSIS. — SUBSTITUTE & SOLVE FOR ξ :

$$u_j^0 = e^{ikh_j\Delta}$$

$$u_j^1 = \xi e^{ikh_j\Delta}$$

USING THE DIFFERENCE FORMULA, WE CAN SIMPLY READ OFF ξ :

$$\xi = 1 + \frac{\delta K}{\Delta^2} \left[2 \cos k\Delta - 2 + (1+\epsilon) \frac{\delta K}{2\Delta^2} (2 \cos 2k\Delta - 8 \cos k\Delta + 6) \right]$$

THIS ONE IS NOT SO EASY TO SKETCH, BUT THERE ARE
 GRAPHS IN examples/diffusion 1/stability/ } SEE
 AND IN " " /stability 2/ } VIEWFOLDS

IT TURNS OUT THAT THE MIDPOINT METHOD IS STABLE
 FOR

$$\delta < \frac{\Delta^2}{K}$$

(TWICE THE MAXIMUM TIMESTEP OF FTCS). WITH $S=2$
 ($\delta = \Delta^2/2K$), THE METHOD HAS VERY HIGH FIDELITY
 COMPARED TO FTCS OR EVEN ITCS WITH THE
 SAME δ .

THOUGH IT WOULD BE LOTS OF WORK TO IMPLEMENT AN IMPLICIT MIDPOINT, WE CAN EVALUATE ITS ANTICIPATED PERFORMANCE WITH VON NEUMANN'S TRICK QUITE PAINLESSLY.

THE DIFFERENCING SCHEME IS LIKE THE EXPLICIT VERSION, JUST SWITCH TIME INDICES $1 \rightarrow 0; \delta \rightarrow -\delta;$

$$u_j^0 = u_j^1 - \frac{\delta k}{\Delta^2} \left[u_{j+1}^1 + u_{j+1}^0 - 2u_j^1 - (1+\epsilon) \frac{\delta k}{2\Delta^2} (u_{j+2}^1 + u_{j+2}^0 - 4u_{j+1}^1 - 4u_{j+1}^0 + 6u_j^1) \right]$$

NO NEED TO SOLVE FOR u_j^1 ; IN FACT, LET

$$u_j^0 = e^{ikj\Delta}, \quad u_j^1 = \xi e^{ikj\Delta}$$

AND WE GET

$$\xi = \frac{1}{1 - \frac{\delta k}{\Delta^2} \left[2\cos k\Delta - 2 - (1+\epsilon) \frac{\delta k}{2\Delta^2} (2\cos 2k\Delta - 8\cos k\Delta + 6) \right]}$$

LIKE ITCS, THIS HAS THE PROPERTIES (FOR A RANGE OF ϵ) THAT:

$$0 < \xi < 1 \quad \forall \delta \quad \text{AND}$$

$\xi(k)$ IS MONOTONICALLY DECREASING $\forall \delta,$

WHICH ARE ALSO SHARED BY THE ANALYTIC SOLN. IF E.G. WE CHOOSE $\epsilon = +0.5$, THEN THE IMPLICIT MIDPOINT IS A MUCH CLOSER IMITATION OF THE ANALYTIC FREQUENCY RESPONSE THAN ITCS.

AGAIN, SEE THE GRAPHS.

COMMENTS REGARDING $\xi(k)$ GRAPHS ON THE WEB & VIDEOS:

- VON NEUMANN'S TRICK ALLOWS US TO EVALUATE TRADEOFFS AMONG SCHEMES WITH MINIMAL EFFORT.
- NO ONE SCHEME IS BEST FOR BOTH SHORT AND LONG TIMESTEPS FOR THE DIFFUSION PROBLEM.
- USING ITCs FOR SEVERAL LONG TIMESTEPS, DOES IT EVEN MATTER THAT THE HIGH FREQUENCY TAILS OF ξ ARE A LITTLE FAT? ξ^N IS STILL A SMALL NUMBER; HIGH FREQUENCIES GO AWAY QUICKLY.
(DISCUSS).
- WHY NOT JUST FFT AND THEN USE THE ANALYTIC FREQUENCY RESPONSE? THIS IS CERTAINLY DONE, BUT THERE ARE COMPLICATIONS:
 - DOESN'T WORK WHEN $K = K(u)$ (NONLINEAR DIFFUSION)
 - DIFFICULT TO IMPLEMENT IN INTERESTING GEOMETRIES, OR WITH SOME TYPES OF BOUNDARY CONDITIONS.

Eigenvalues as a function of k for the linear diffusion equation with various time stepping schemes. Dimensionless timestep $n = 2 \cdot dt \cdot K / dx^2$, where K is the diffusion constant and dx is the grid spacing. In these units, the maximum stable FTCS timestep is:

$$n = 1$$

To plot wavenumber on the x-axis, I define

$$k = x$$

Forward time, centered space (FTCS):

$$\blacksquare y = 1 + n (\cos k - 1)$$

Midpoint method with perturbed halfstep, $dt \cdot (1+d)/2$:

$$d = -0.5$$

$$\blacksquare y = 1 + n \left(\cos k - 1 + \frac{(1+d)n}{4} (\cos 2k - 4 \cos k + 3) \right)$$

Implicit time, centered space (ITCS):

$$\blacksquare y = \frac{1}{1 + n (1 - \cos k)}$$

Implicit midpoint method. This time, the intermediate timestep is $(1+c)/2$:

$$c = 0.5$$

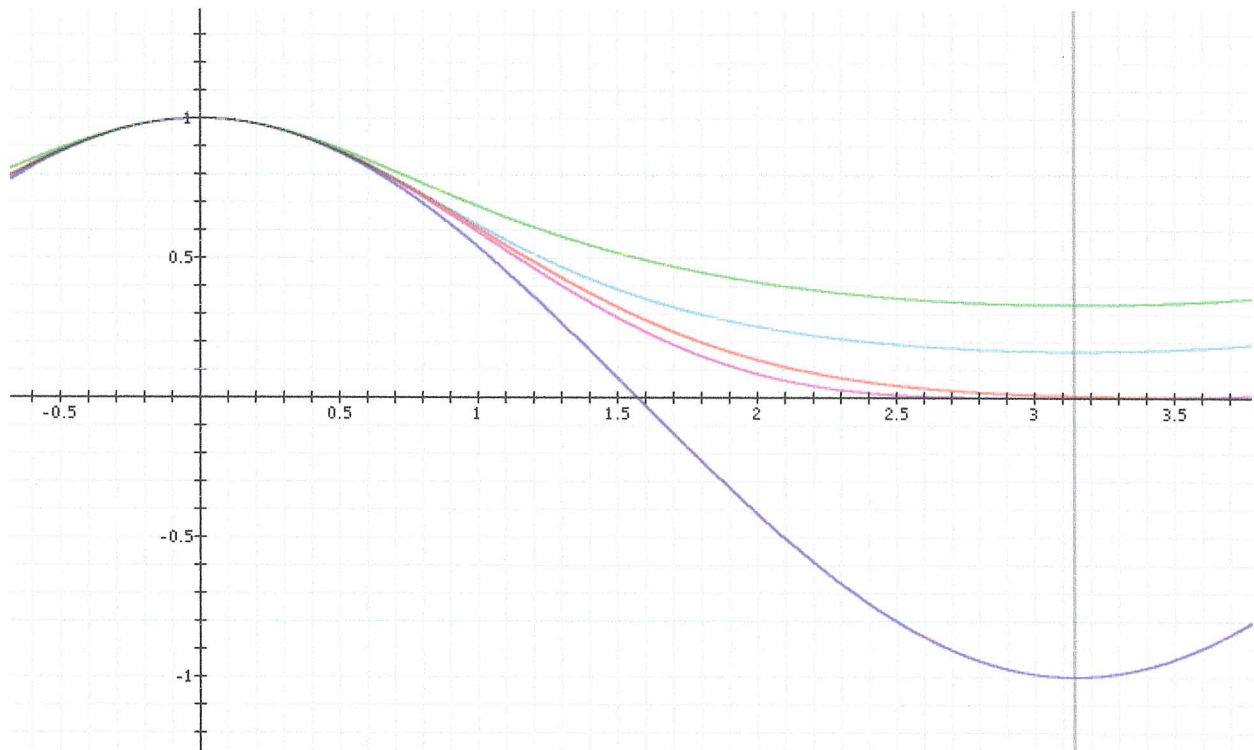
$$y = \frac{1}{1 - n \left(\cos k - 1 - \frac{(1+c)n}{4} (\cos 2k - 4 \cos k + 3) \right)}$$

Analytic result:

$$y = e^{-\frac{nk^2}{2}}$$

Nyquist Frequency:

$$x = \pi$$



Eigenvalues as a function of k for the linear diffusion equation with various time stepping schemes. Dimensionless timestep $n = 2 \cdot dt \cdot K / dx^2$, where K is the diffusion constant and dx is the grid spacing. This time we use a very long timestep (compare diffusion.c):

$$n = 100$$

To plot wavenumber on the x-axis, I define

$$k = x$$

Forward time, centered space (FTCS):

$$\blacksquare y = 1 + n (\cos k - 1)$$

Midpoint method with perturbed halfstep, $dt \cdot (1+d)/2$:

$$d = -0.5$$

$$\blacksquare y = 1 + n \left(\cos k - 1 + \frac{(1+d)n}{4} (\cos 2k - 4 \cos k + 3) \right)$$

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