

OPERATOR SPLITTING

(§ 19.3)

①

- SUPPOSE I WANT TO SOLVE

$$\frac{\partial u}{\partial t} = K \nabla^2 u$$

$$\nabla^2 \psi = 0$$

⋮

HOW DO I GENERALIZE MY SCHEME TO
MULTIPLE SPATIAL DIMENSIONS?

- HOW DO I TREAT AN EQUATION OF MIXED TYPE:

$$\frac{\partial \vec{u}}{\partial t} = \underbrace{-(\vec{u} \cdot \nabla) \vec{u}}_{\substack{\uparrow \\ \text{HYPERBOLIC} \\ \text{(EXPLICIT)}}} + \mu \underbrace{\nabla^2 \vec{u}}_{\substack{\uparrow \\ \text{PARABOLIC} \\ \text{(IMPLICIT)}}$$

CONSIDER:

$$\frac{\partial u}{\partial t} = \mathcal{L} u$$

EXPRESS AS A SUM OF SUB-PARTS:

$$\mathcal{L} u = \mathcal{L}_1 u_1 + \mathcal{L}_2 u_2 + \dots + \mathcal{L}_m u_m$$

(2)

DIVIDE TIME STEP δ INTO m SUB-PARTS
(EACH IS A FULL TIMESTEP δ ON $\frac{1}{m}$ OF L !!!)

FOR EACH L_i , WE NEED AN "UPDATING SCHEME" \mathcal{U}_i .
" \mathcal{U}_i " IS THE SORT OF SCHEME YOU'D USE FOR THE SIMPLER PDE!

$$\frac{\partial u}{\partial t} = L_i u \quad \longleftrightarrow \quad u_j^{n+1} = \mathcal{U}_i(u^n, \delta)$$

[PDE]

SEQUENCE OF SUB-STEPS TO GET ALL OF L :

$$u^{n+\frac{1}{m}} = \mathcal{U}_1(u^n, \delta) \quad \leftarrow \text{FULL } \delta \text{ OVER PART OF } L.$$

$$u^{n+\frac{2}{m}} = \mathcal{U}_2(u^{n+\frac{1}{m}}, \delta).$$

⋮

$$u^{n+1} = \mathcal{U}_m(u^{n+\frac{m-1}{m}}, \delta).$$

WHY NOT:

$$u^{n+1} = u^n + \sum_{i=1}^m \left[\mathcal{U}_i(u^n, \delta) - u^n \right]$$

DISCUSS. (AS FAR AS I KNOW, THE ABOVE MIGHT WORK FINE)

(OPTIONAL)

2.5

ANOTHER OPERATOR SPLITTING APPROACH:

$$\frac{\partial u}{\partial t} = L u$$

$$L u \rightarrow L_1 u + L_2 u + \dots + L_m u$$

LET \mathcal{D}_i BE AN UPDATING SCHEME FOR ALL OF L
THAT IS STABLE ONLY FOR L_i ~~NOW DO m~~
~~PARTIAL TIMESTEPS (δ/m) FOR THE FULL L :~~

$$\frac{\partial u}{\partial t} = L u \leftrightarrow u^{n+1} = \mathcal{D}_i(u^n, \delta) \quad \forall i$$

NOW DO m PARTIAL TIMESTEPS δ/m :

$$u^{n+\frac{1}{m}} = \mathcal{D}_1(u^n, \delta/m)$$

$$u^{n+\frac{2}{m}} = \mathcal{D}_2(u^{n+\frac{1}{m}}, \delta/m)$$

$$u^{n+1} = \mathcal{D}_m(u^{n+\frac{(m-1)}{m}}, \delta/m)$$

THIS PROCEDURE IS USUALLY STABLE.

LESSON: THERE IS MORE THAN ONE WAY
TO DO IT! (TIMTOWTDI)

EXAMPLE

ALTERNATING DIRECTION IMPLICIT (ADI) METHOD

SUPPOSE WE HAVE A DIFFUSION EQN IN 2D:

$$\frac{\partial \Phi}{\partial t} = k \nabla^2 \Phi \rightarrow k \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right)$$

X-IMPLICIT TIMESTEP:

$$\star \Phi_{j,k}^{n+1/2} = \Phi_{j,k}^n + \frac{\delta k}{2 \Delta^2} \left(\Phi_{j+1,k}^{n+1/2} + \Phi_{j-1,k}^{n+1/2} - 2\Phi_{j,k}^{n+1/2} \right) \leftarrow \text{X-IMPLICIT}$$

$$\frac{\delta k}{2 \Delta^2} \left(\Phi_{j,k+1}^n + \Phi_{j,k-1}^n - 2\Phi_{j,k}^n \right) \leftarrow \text{Y-EXPLICIT}$$

$$\begin{aligned} \Rightarrow -\frac{\delta k}{2 \Delta^2} \Phi_{j-1,k}^{n+1/2} + \left(1 + \frac{\delta k}{\Delta^2} \right) \Phi_{j,k}^{n+1/2} - \frac{\delta k}{2 \Delta^2} \Phi_{j+1,k}^{n+1/2} \\ = \frac{\delta k}{2 \Delta^2} \Phi_{j,k+1}^n + \left(1 - \frac{\delta k}{\Delta^2} \right) \Phi_{j,k}^n + \frac{\delta k}{2 \Delta^2} \Phi_{j,k-1}^n \end{aligned}$$

WHICH HAS THE FORM

$$A \vec{\Phi}^{n+1/2} = \vec{\alpha}$$

↑ TRIDIAGONAL, EASILY INVERTED

$$\boxed{\vec{\Phi}^{n+1/2} = A^{-1} \vec{\alpha}(\vec{\Phi}^n)}$$

WE SIMILARLY DO A Y-IMPLICIT TIMESTEP,

$$\boxed{\vec{\Phi}^{n+1} = B^{-1} \vec{\beta}(\vec{\Phi}^{n+1/2})}$$

COMBINING THE TWO HALF-TIMESTEPS YIELDS A STABLE (SO I CLAIM!) DIFFERENCING SCHEME.

IN-CLASS EXERCISE

(4)

USE VON NEUMANN'S TECHNIQUE TO EVALUATE THE STABILITY OF ADI IN 2D.

① LET $\Phi_{j,k}^n = e^{i(Jj + Kk)}$
ASSUME $\Phi_{j,k}^{n+1/2} = \xi_1 e^{i(Jj + Kk)}$ } (J & K DIMENSIONLESS WAVENUMBERS)
 $0 < J < \pi$
 $0 < K < \pi$

TRY OUT ON EQN. ★:

$$\xi_1 = 1 + \frac{\delta}{2} \frac{\kappa}{\Delta^2} \xi_1 (e^{iJ} + e^{-iJ} - 2) + \frac{\delta}{2} \frac{\kappa}{\Delta^2} (e^{iK} + e^{-iK} - 2)$$

← KAPPA, NOT K!

$$\xi_1 = \frac{1 + \frac{\delta \kappa}{\Delta^2} (\cos K - 1)}{1 - \frac{\delta \kappa}{\Delta^2} (\cos J - 1)}$$

② FOR THE SECOND HALF-TIMESTEP, WE INFER BY SYMMETRY AN EIGENVALUE

$$\xi_2 = \frac{1 + \frac{\delta \kappa}{\Delta^2} (\cos J - 1)}{1 - \frac{\delta \kappa}{\Delta^2} (\cos K - 1)}$$

③ TOTAL TIMESTEP HAS EIGENVALUE

$$\xi = \xi_1 \xi_2 = \frac{(1-c)^2 + (1-c)c(\cos J + \cos K) + c^2 \cos J \cos K}{(1+c)^2 - (1+c)c(\cos J + \cos K) + c^2 \cos J \cos K}$$

WHERE $c \equiv \frac{\delta \kappa}{\Delta^2}$, DIMENSIONLESS TIMESTEP

THIS IS STABLE $\forall \delta, \forall J, \forall K$.

→ GRAPHING CALCULATOR FILE examples/diffusionADI/adi.gcf