

# THE TOPOLOGICAL NATURE OF THE PARKER MAGNETOSTATIC THEOREM<sup>a)</sup>

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The two-plate initial boundary-value problem of Parker is reviewed, treating the relaxation of a 3D magnetic field prescribed with an arbitrary topology to a terminal force-free field in a cold, viscous, electrically perfect fluid-conductor. Anchored by their foot-points at the perfectly conducting rigid plates, the relaxing field preserves its topology. The Parker Magnetostatic Theorem states that for most prescribed field topologies, the terminal field must embed current sheets. The elements of this Theorem are examined to relate this initial boundary-value problem to (i) the variational problem for a force-free field of a given topology and (ii) the direct construction of a force-free field in terms of its pair of Euler flux functions. Insights are presented on the Theorem as the compelling basis of the Parker theory of solar coronal heating.

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## I. INTRODUCTION

The fundamental Magnetostatic Theorem of Eugene N. Parker states that a 3D magnetic field of complex topology, frozen into an electrically perfect fluid-conductor, generally attains force equilibrium by spontaneously embedding infinitesimally thin current-sheets (CSs) across which the field is tangentially discontinuous. In the absence of resistivity, the discrete electric current flowing in a magnetic tangential discontinuity (TD), given by Ampere's law, arises naturally. Magnetic flux surfaces move as fluid surfaces. In a 3D evolution to equilibrium, two flux systems separate in space can readily push an intervening third flux system completely out of their way to make direct contact at a TD. The crux of the Theorem is that *most equilibrium fields* must embed TDs by the nonlinear action of the anisotropic Lorentz force under the frozen-in condition. That is, equilibria embedding TDs are the rule rather than the exception. This concept-transforming property was discovered by Parker<sup>1</sup> 50 years ago. The basis for it and its implications were carefully laid out in a monograph<sup>2</sup> with extensive references, hereafter referred to as the monograph.

The Theorem originated in the Parker theory of coronal heating under the condition of near-perfect electrical conductivity<sup>1-6</sup>. The adjective near-perfect is taken in the hydromagnetic sense of a large Lundquist number  $\mathcal{L}_\eta = \frac{U_0 L_0}{\eta}$  in a macroscopic fluid flow of characteristic speed  $U_0$ , length scale  $L_0$  and electrical resistivity  $\eta$ . Treating the corona as fully ionized hydrogen at temperature  $T = 10^6 K$  dominated by a  $\sim 10 G$  field, the Spitzer resistivity gives  $\eta \sim 0.5 \times 10^{13} T^{-3/2} = 5 \times 10^3 cm^2 s^{-1}$ . Setting  $U_0 \sim 500 km s^{-1}$ , a characteristic coronal Alfvén speed, we see that the corona behaves like a perfect conductor with  $\mathcal{L}_\eta \geq 10^{10}$  down in scale lengths to that of the thermal proton mean-free-path  $L_{mfp} \sim 100 km$ . When not disrupted by high energy flares and Corona Mass Ejections, the low- $\beta$  corona yields to its  $\sim 10 G$  field and evolves quasi-statically. The evolution is driven by the heavy photosphere slowly moving at speeds  $\sim 0.5 km s^{-1}$  compared to the coronal Alfvén speeds  $\geq 500 km s^{-1}$  at which the corona adjusts dynamically to remain in its large-scale near-equilibrium state.

Through the thin ( $10^4 km \approx 3 \times 10^{-2} R_\odot$ ) chromosphere, temperature rises steeply from  $\sim 6 \times 10^3 K$  at the photosphere to the million degree temperature in the low corona, as the near hydrostatic density falls some 8 orders of magnitude. This gravitationally-bound transition layer is rich with time-dependent radiative and hydromagnetic processes nowadays

observed with impressive spectral, spatial and temporal resolutions<sup>7</sup>. We will not treat the chromosphere, interesting in its own right, except to state that, through its photospheric magnetic fields extend into the corona, delivering an ever present Poynting flux at a rate adequate to account for the estimated global loss of  $10^{28} \text{ erg s}^{-1}$  in thermal X-ray from the corona<sup>2,3,8</sup>. The essence of the heating problem is concerned with how magnetic energy stored in the field under high electrical conductivity might nevertheless be resistively dissipated as a source of heating.

The Theorem makes the point that current sheets form irrepressibly in the coronal field under the condition  $\mathcal{L}_\eta \gg 1$ , only to have the condition break down locally wherever a current sheet thins towards zero width. The thinning sheet then dissipates via magnetic reconnection taking place with the otherwise negligible resistivity<sup>9,10</sup>. The dissipation locally heats the corona and, by its removal of the current sheet, the high-conductivity condition  $\mathcal{L}_\eta \gg 1$  is locally restored. Under the Theorem, returning the field to the perfect-conductivity regime merely sets the stage for further formation of current sheets and their subsequent dissipation. These dissipative events explain the ubiquitous, intermittent, small-scale nano-flares observed in high-resolution coronal X-ray observations<sup>2,3</sup>.

The present paper focuses on the Theorem as a basic MHD property while keeping in mind its fundamental role in coronal heating. Section II presents a constructive analysis of the two-plate initial boundary-value problem introduced by Parker<sup>1</sup>, treating the relaxation of a field to a terminal force-free state in a cold, viscous, perfectly conducting fluid<sup>11</sup>. A variational description of the terminal state presents an instructive comparison with the force-free field as an analytical solution to the nonlinear, coupled PDEs describing a global pair of Euler flux-functions. The topological properties of the frozen-in fields are central to the Theorem. In particular, what is meant by *most equilibrium fields* will be defined with conceptual precision. Section III presents a concluding discussion, pointing out several basic implications previously not known or appreciated and offering a view that there is interesting work to do on the low corona as a quasi-steadily driven,  $\beta \ll 1$ ,  $\mathcal{L}_\eta \gg 1$  turbulent hydromagnetic atmosphere.

## II. THE PARKER TWO-PLATE PROBLEM

The two-plate problem seeks a time-dependent solution for a field  $\mathbf{B}$  in the unbounded fluid between two parallel, rigid, perfectly-conducting Cartesian boundaries at  $z = 0, L$ . The field is unidirectional, streaming from one plate to the other where the magnetic foot-points are rigidly anchored. We treat physically the simplest case of a cold, viscous, perfectly conducting fluid described by

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu_1 \nabla^2 \mathbf{v} + \nu_2 \nabla (\nabla \cdot \mathbf{v}), \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (3)$$

in standard notations, subject to the boundary-initial conditions:

$$z = 0, L, \quad \mathbf{v} = 0, \quad B_z = B_0 > 0, \quad (4)$$

$$R = (x^2 + y^2)^{1/2} \rightarrow \infty, \quad \mathbf{v} \rightarrow 0, \quad \mathbf{B} \rightarrow \mathbf{B}_0 = B_0 \hat{\mathbf{z}}, \quad (5)$$

$$t = 0, \quad \rho = \rho_0, \quad \mathbf{v} = 0, \quad \mathbf{B} = \mathbf{B}_\tau(\mathbf{r}), \quad (6)$$

where  $\mathbf{r} = (x, y, z)$ ;  $\nu_1$  and  $\nu_2$  are constant coefficients of viscosity;  $\rho_0$  is a uniform initial density;  $\mathbf{B}_0$  is a uniform field in the  $z$  direction, and,  $\mathbf{B}_\tau$  is an arbitrarily prescribed initial field subject to boundary conditions (4) and (5).

A few comments are in order. Boundary conditions (4) follow from induction equation (3) under the Maxwell requirement for the tangential component of the electric field  $\mathbf{E} = -\frac{1}{c} \mathbf{v} \times \mathbf{B}$  to vanish at  $z = 0, L$  as rigid, perfectly conducting boundaries. Independent of induction equation (3), viscosity also demands that  $\mathbf{v} = 0$  at  $z = 0, L$ .

Figure 1 is a sketch of two initial fields, the uninteresting case of a uniform initial field  $\mathbf{B}_\tau \equiv \mathbf{B}_0$  in  $0 < z < L$  and a case of a continuous  $\mathbf{B}_\tau(\mathbf{r})$  in a significantly twisted state. The former is a trivial solution to the two-plate problem with  $\mathbf{v} \equiv 0$ . Following Parker's original treatment, we limit  $\mathbf{B}_\tau$  to be created from the uniform field  $\mathbf{B}_0$  in a specific manner. Starting with  $\mathbf{B}_0$ , relax the rigidity of boundary  $z = 0$ . Then, carry out a continuous, tangential displacement of the magnetic foot-points on  $z = 0$  and spatially continue the displacement into the interior  $0 < z < L$ , while holding  $z = L$  rigid. Taking the foot-point displacement to be incompressible and spatially bounded in the  $x$  and  $y$  directions ensures

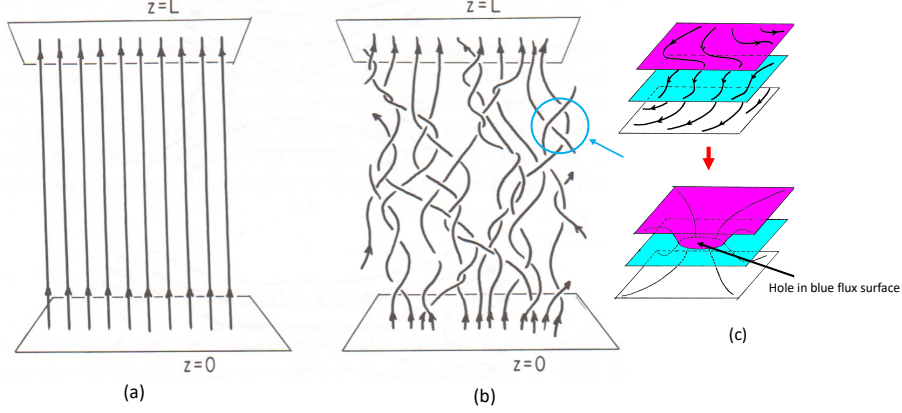


FIG. 1. The fields (a)  $\mathbf{B}_0$  and (b)  $\mathbf{B}_\tau(\mathbf{r})$  described in the text. The sketch (c) of two flux surfaces pressed into contact through a hole punched into a third intervening flux surface, discussed in Section II.C.

boundary conditions (4) and (5), respectively. Identify  $\mathbf{B}_\tau$  with the deformed field obtained and restore  $z = 0$  to rigidity.

Define  $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$  and obtain the free magnetic energy

$$\mathcal{E} = \frac{1}{8\pi} \int_{0 < z < L} [\mathbf{B} \cdot \mathbf{B} - \mathbf{B}_0 \cdot \mathbf{B}_0] dV = \frac{1}{8\pi} \int_{0 < z < L} \mathbf{b} \cdot \mathbf{b} dV, \quad (7)$$

where  $dV = dx dy dz$ . A product term  $\mathbf{b} \cdot \mathbf{B}_0$  in the integrand has no contribution by virtue of the solenoidal condition  $\nabla \cdot \mathbf{b} = 0$  and boundary condition (4), the latter implying  $b_z = 0$  at  $z = 0, L$ . Boundary condition (5) in terms of  $\mathbf{b} \rightarrow 0$  as  $R \rightarrow \infty$  is imposed for a finite free energy  $\mathcal{E}$ . The governing equations (1)-(3) subject to the boundary conditions imply that the sum of the total kinetic energy and free magnetic energy  $\mathcal{E}$  must decrease monotonically by viscous dissipation<sup>11</sup>,

$$\frac{d}{dt} \int_{0 < z < L} \left[ \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \frac{1}{8\pi} \mathbf{b} \cdot \mathbf{b} \right] dV = - \int_{0 < z < L} \left[ \nu_1 (\nabla \times \mathbf{v})^2 + \nu_2 (\nabla \cdot \mathbf{v})^2 \right] dV. \quad (8)$$

It follows that as  $t \rightarrow \infty$ ,  $\mathbf{v} \rightarrow 0$ , as the field relaxes to the force-free field  $\mathbf{B}_f(\mathbf{r}) = \lim_{t \rightarrow \infty} \mathbf{B}(\mathbf{r}, t)$  described by

$$(\nabla \times \mathbf{B}_f) \times \mathbf{B}_f = 0, \quad (9)$$

$$\nabla \cdot \mathbf{B}_f = 0. \quad (10)$$

If the terminal field  $\mathbf{B}_f(\mathbf{r})$  is analytic, then the vanishing of the Lorentz force implies

$$\nabla \times \mathbf{B}_f = \alpha \mathbf{B}_f, \quad (11)$$

$$\mathbf{B}_f \cdot \nabla \alpha = 0, \quad (12)$$

respectively expressing Ampere's law for a field-aligned current density and the solenoidal condition requiring the proportionality function  $\alpha$  to be constant along each field line<sup>12,13</sup>. The function  $\alpha$  defining the current density must (i) yield a field solution  $\mathbf{B}_f(\mathbf{r})$  that meets boundary conditions (4) and (5), and (ii) be demonstrably the result of a continuous deformation of the initial field  $\mathbf{B}_\tau(\mathbf{r})$  by the viscous flow.

The two-plate problem for  $\mathbf{B}_f(\mathbf{r})$  is topological in nature, not a traditional boundary value problem. Let us use the textbook definition of topology encompassing the properties of a given geometric object that are invariant under all continuous deformation of the object. Topological properties are formidable to described explicitly, e.g., the mutual linkages among closed tubes of magnetic flux or, in the case of the two-plate problem, the mutual field-line mapping of the foot-points fixed on  $z = 0, L$  and the interweaving among the field lines, both invariant under the frozen-in condition. We avoid such formidable explicit descriptions by the conceptual sufficiency of knowing that the two fields  $\mathbf{B}_\tau(\mathbf{r})$  and  $\mathbf{B}_f(\mathbf{r})$  are topologically equivalent if and only if one is continuously deformable into the other.

A continuum of scalar functions  $\alpha$  exists for which  $\mathbf{B}_\tau(\mathbf{r})$  and  $\mathbf{B}_f(\mathbf{r})$  are topologically equivalent analytical fields. The Theorem makes its first essential point here, that a topologically-equivalent analytical  $\mathbf{B}_f(\mathbf{r})$  generally does not exist for an arbitrarily prescribed  $\mathbf{B}_\tau(\mathbf{r})$ . Equation (12) is a hyperbolic PDE for  $\alpha$ , with the field lines as the characteristics that, in general, do not *a priori* allow  $\mathbf{B}_f(\mathbf{r})$  to be continuous everywhere; see Chapter 3 of the monograph.

To account for TDs in the field  $\mathbf{B}_f(\mathbf{r})$ , we formally generalize (11) to the form:

$$\nabla \times \mathbf{B}_f = \alpha \mathbf{B}_f + \sum_j \mathcal{C}_j (\mathcal{S}_j). \quad (13)$$

as a statement of Ampere's law, where  $\mathcal{C}_j$  are a set of discrete current flowing in their respective TDs located on flux surfaces  $\mathcal{S}_j$ . By  $\mathcal{C}_j$  being discrete it is meant the "profile" of the current density across  $\mathcal{S}_j$  is a Dirac  $\delta$ -function, the current density being unbounded but carrying a finite integrated current<sup>14,15</sup>. This Ampere equation is coupled to equation (12) describing field-aligned current density in the continuous part of the magnetic field.

The discrete currents  $\mathcal{C}_j$  flowing in surfaces  $\mathcal{S}_j$  cannot be arbitrarily inserted into the otherwise continuous force-free field. Decompose the Lorentz force into the sum of magnetic-

tension and isotropic-pressure forces,

$$\frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left( \frac{\mathbf{B} \cdot \mathbf{B}}{8\pi} \right). \quad (14)$$

The integral of the Lorentz force across a TD surface  $\mathcal{S}_j$  has no contribution from the tension force, with the net force on  $\mathcal{S}_j$  given by the difference between the pressure forces on its two sides. Therefore, the surface  $\mathcal{S}_j$  in force equilibrium requires the magnetic pressure to be continuous across it. That is, the TDs embedded in  $\mathbf{B}_f(\mathbf{r})$  are in equilibrium with the tangential field rotating an abrupt angle at constant amplitude across each TD. This TD equilibrium condition is subject to the requirement that  $\mathbf{B}_f(\mathbf{r})$  given by the Ampere's law is topologically equivalent to  $\mathbf{B}_\tau(\mathbf{r})$ .

The Theorem is stronger than stated above. For *most* prescribed  $\mathbf{B}_\tau(\mathbf{r})$ , its terminal field  $\mathbf{B}_f(\mathbf{r})$  must embed equilibrium TDs. A variational description of  $\mathbf{B}_f(\mathbf{r})$  offers a useful perspective.

### A. The variational two-plate problem

Consider the deformation of the frozen-in field in terms of the Lagrangian fluid displacements under mass conservation and induction equations (2) and (3). The end result of any continuous deformation of the fluid is a reversible, one-to-one Lagrangian transformation<sup>16</sup>

$$\mathbf{r} = \mathbf{r}(\mathbf{r}'), \quad (15)$$

$$L_{ij}(\mathbf{r}, \mathbf{r}') = \frac{\partial x_i}{\partial x'_j}, \quad (16)$$

$$|L| = \epsilon_{ijk} \frac{\partial x_1}{\partial x'_i} \frac{\partial x_2}{\partial x'_j} \frac{\partial x_3}{\partial x'_k} \neq 0, \quad (17)$$

describing the Cartesian location of each fluid element at  $\mathbf{r} = (x_1, x_2, x_3) \equiv (x, y, z)$  displaced from its previous Cartesian location at  $\mathbf{r}' = (x'_1, x'_2, x'_3) \equiv (x', y', z')$ . Continuity of deformation means that the infinitesimal neighborhood of every fluid element in  $\mathbf{r}'$ -space is transformed onto an infinitesimal neighborhood of the fluid element in  $\mathbf{r}$ -space. Thus, different histories of continuous deformations are topologically equivalent if they produce at the end the same Lagrangian transformation of the fluid elements.

Equations (2) and (3) are equivalent to the Lagrangian displacements of fluid and field

$$\rho(\mathbf{r}) = \frac{\rho'(\mathbf{r}')}{|L|}, \quad (18)$$

$$B_j(\mathbf{r}) = \frac{L_{ij}}{|L|} B'_i(\mathbf{r}'), \quad (19)$$

the prime denoting variables in the Cartesian  $\mathbf{r}'$ -space. The flux surfaces of the field deform as fluid surfaces. Thus, any pair of flux surfaces of  $\mathbf{B}(\mathbf{r})$  transforms into a corresponding pair of flux surfaces of  $\mathbf{B}'(\mathbf{r}')$  sandwiching the same "frozen" fluid between them.

In the two-plate problem, any given continuous field in  $0 < z < L$ , with foot-points fixed at the boundaries, defines a vector-function space  $\mathcal{B}(\tau)$  of topologically-equivalent continuous fields, now interpreting  $\tau$  to denote the common topology of the fields. In other words, any two fields in  $\mathcal{B}(\tau)$  are continuously deformable one into the other, related by a reversible Lagrangian transformation (16). Suppose  $\mathcal{B}(\tau)$  contains a unique field with the lowest field energy  $\mathcal{E}$  in  $\mathcal{B}(\tau)$ . Then, this field is the terminal field  $\mathbf{B}_f(\mathbf{r})$  of the time-dependent solution to the two-plate problem. That is, in the space  $\mathcal{B}(\tau)$ , each field  $\mathbf{B}_\tau(\mathbf{r})$  as the initial state is connected by a viscous evolutionary path to the field  $\mathbf{B}_f(\mathbf{r})$ .

We thus arrive at the variational problem,

$$\delta\mathcal{E} \equiv \frac{1}{8\pi} \delta \int_{0 < z < L} \mathbf{b} \cdot \mathbf{b} \, dV = 0, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b} \in \mathcal{B}(\tau), \quad (20)$$

that identifies  $\mathbf{B}_f(\mathbf{r})$  as an extremum in  $\mathcal{E}$ . Then, force-free equations (11) and (12) are the extremizing Euler-Lagrangian equations to be solved for  $\mathbf{B}_f(\mathbf{r})$  subject to (i) boundary conditions (4) and (5), and (ii) the condition that  $\mathbf{B}_f(\mathbf{r}) \in \mathcal{B}(\tau)$ , i.e.,  $\mathbf{B}_f(\mathbf{r})$  is topologically equivalent to a given  $\mathbf{B}_\tau(\mathbf{r})$ .

The Theorem states that, in general, the terminal field  $\mathbf{B}_f(\mathbf{r})$  may contain TDs, in which case,  $\mathbf{B}_f(\mathbf{r}) \notin \mathcal{B}(\tau)$  and the variational problem (20) defined over the space  $\mathcal{B}(\tau)$  does not have a solution. This is an intrinsic feature of variational calculus<sup>17</sup>. Since  $\mathcal{E}$  is bounded from below, it unquestionably has a minimum value in  $\mathcal{B}(\tau)$ , whereas that minimum may not be realized by any field in  $\mathcal{B}(\tau)$ . In practice the minimum can be determined by an evolutionary path in  $\mathcal{B}(\tau)$  of a monotonically decreasing  $\mathcal{E}$ , such as defined by a viscosity-driven time-dependent solution to the two-plate problem. If  $\mathbf{B}_f(\mathbf{r})$  contains TDs, the  $\mathcal{E}$ -minimizing path can approach the minimum value as close as desired without leaving  $\mathcal{B}(\tau)$ . Generally, the space  $\mathcal{B}(\tau)$  is mathematically not compact, in the analogous sense of a partially open interval  $0 \leq x < 1$ . In the latter, an infinite series of points can converge as close as desired on the end point  $x = 1$  without leaving the interval, despite the fact that this end point does not belong to the interval. Whereas, an infinite series can converge and arrive at the other endpoint  $x = 0$  which belongs to the interval.



Specifying the space  $\mathcal{B}(\tau)$  completes the statement of the variational problem (20), without which the extremum condition  $\delta\mathcal{E} = 0$  is neither defined nor meaningful. The analytical force-free field of a topology  $\tau$  satisfying equations (11) and (12) is an instance of the variational problem possessing a solution. But, this field belongs to a topologically restricted set, for there is the other distinct set of fields in topologies requiring the presence of equilibrium TDs. The existence of the latter is the reason for the general absence of a solution to the variational problem defined by specific spaces  $\mathcal{B}(\tau)$ . The TD-bearing force-free fields are not discontinuous in an arbitrary manner but are identified by their respective spaces  $\mathcal{B}(\tau)$ . They satisfy the Euler-Lagrangian equations as weak solutions, their discontinuities satisfying force balance in the integral sense analogous to the hyperbolic equations of compressible hydrodynamics<sup>18</sup>. Hydrodynamic shocks are weak solutions that satisfy the conservative PDEs in the integral sense of preserving the conserved quantities across the shocks.

The minimum-energy force-free field of any given topology  $\tau$  must physically exist under the frozen-in condition, and this is ensured by including the weak solutions to the variational problem (20). Here we arrive at the practical problem of how a given invariant topology is to be implemented in the direct construction of a force-free field.

## B. Euler flux-function construction of force-free fields

Let us assume analyticity and represent a field by a pair of Euler flux functions<sup>16</sup>,

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \nabla U \times \nabla V, \\ &= \frac{\partial(U, V)}{\partial(y, z)} \hat{\mathbf{x}} + \frac{\partial(U, V)}{\partial(z, x)} \hat{\mathbf{y}} + \frac{\partial(U, V)}{\partial(x, y)} \hat{\mathbf{z}}, \end{aligned} \quad (21)$$

automatically satisfying solenoidal condition (10). The representation implies  $\mathbf{B} \cdot \nabla U = \mathbf{B} \cdot \nabla V = 0$ , showing that the level surfaces of  $U$  and  $V$  are magnetic flux surfaces that mutually intersect along field lines. Thus, each field line is a line of constant  $U$  and  $V$ . This geometric relationship also applies to the field-aligned current density of a force-free field, for which  $(\nabla \times \mathbf{B}) \cdot \nabla U = (\nabla \times \mathbf{B}) \cdot \nabla V = 0$ , reducible to

$$\nabla \cdot (\mathbf{B} \times \nabla U) = \nabla \cdot [(\nabla U \times \nabla V) \times \nabla U] = 0, \quad (22)$$

$$\nabla \cdot (\mathbf{B} \times \nabla V) = \nabla \cdot [(\nabla U \times \nabla V) \times \nabla V] = 0. \quad (23)$$

That is, force-free equation (9) is equivalent to the two coupled second-order PDEs for  $(U, V)$ ,

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla U - |\nabla U|^2 \nabla V] = 0, \quad (24)$$

$$\nabla \cdot [(\nabla U \cdot \nabla V) \nabla V - |\nabla V|^2 \nabla U] = 0, \quad (25)$$

to be solved subject to boundary conditions on  $(U, V)$ .

In terms of  $(U, V)$ , induction equation (3) takes the form

$$\frac{\partial U}{\partial t} + \mathbf{v} \cdot \nabla U = 0, \quad (26)$$

$$\frac{\partial V}{\partial t} + \mathbf{v} \cdot \nabla V = 0, \quad (27)$$

describing each fluid element carrying a pair of unchanging values of  $(U, V)$  in any velocity  $\mathbf{v}$ . Consider the displacement  $T_{i,j}(\mathbf{r}, \mathbf{r}')$  of a fluid element from a previous location  $\mathbf{r}'$  to the present location  $\mathbf{r}$ . Then, induction equations (26) and (27) relate  $\mathbf{B}'(\mathbf{r}') = \nabla' U' \times \nabla' V'$  to  $\mathbf{B}(\mathbf{r}) = \nabla U \times \nabla V$  by

$$[U(\mathbf{r}), V(\mathbf{r})] \equiv [(U'(\mathbf{r}'), V'(\mathbf{r}'))]. \quad (28)$$

Flux-surfaces are well defined locally but are not necessarily extendable globally. For example, a 3D field wholly contained in a fluid domain may have infinitely long, volume filling, ergodic field lines that anywhere do lie on locally well-defined flux surfaces, but these flux surfaces cannot be extended globally. Field representation (21) is convenient if the flux functions  $(U, V)$  are globally defined, which is the case for the two-plate problem where the initial field  $\mathbf{B}_\tau(\mathbf{r})$  is prepared by a continuous deformation of the uniform field  $\mathbf{B}_0$ .

The flux surfaces of  $\mathbf{B}_0 = \nabla U_0 \times \nabla V_0 = B_0$  with  $[U_0, V_0] = \sqrt{B_0} [x, y]$  are globally defined, two families of Cartesian planes of constant  $x$  and constant  $y$ . The field  $\mathbf{B}_\tau(\mathbf{r}) = \nabla U_\tau \times \nabla V_\tau$  is prepared from  $\mathbf{B}_0$  by an arbitrarily chosen, continuous fluid displacement  $T_{i,j}^\tau(\mathbf{r}, \mathbf{r}')$ . The flux surfaces of  $\mathbf{B}_\tau(\mathbf{r})$  are thus also globally defined. The fields topologically equivalent to the prepared  $\mathbf{B}_\tau(\mathbf{r})$  define the space  $\mathcal{B}(\tau)$ . Therefore, the two-plate initial-boundary value problem seeks the terminal force-free field  $\mathbf{B}_f(\mathbf{r}) = \nabla U_f \times \nabla V_f$  as a solution to the force-free equations (24) and (25) for  $[U, V] = [U_f, V_f]$ , subject to boundary conditions

$$z = 0, \quad [U, V] = [U_\tau(x, y, 0), V_\tau(x, y, 0)], \quad (29)$$

$$z = L, \quad [U, V] = [U_\tau(x, y, L), V_\tau(x, y, L)] = \sqrt{B_0} [x, y], \quad (30)$$

$$R = (x^2 + y^2)^{1/2} \rightarrow \infty, \quad [U, V] \rightarrow \sqrt{B_0} [x, y], \quad (31)$$

and subject to the *additional* requirement that  $\mathbf{B}_f(\mathbf{r})$  and  $\mathbf{B}_\tau(\mathbf{r})$  are topologically equivalent.

There are two Lagrangian transformations in this construction of the force-free field  $\mathbf{B}_f(\mathbf{r})$ , the first being  $T_{i,j}^\tau(\mathbf{r}, \mathbf{r}')$  that prepares  $\mathbf{B}_\tau(\mathbf{r})$  from  $\mathbf{B}_0$ . The other transformation denoted by  $T_{i,j}^f(\mathbf{r}, \mathbf{r}')$  is defined by the field solution  $\mathbf{B}_f(\mathbf{r})$  as a solution of the above boundary-value problem, which is required to be continuously deformable from the prepared  $\mathbf{B}_\tau(\mathbf{r})$  represented by a relationship

$$[U_f(\mathbf{r}), V_f(\mathbf{r})] \equiv [(U_\tau'(\mathbf{r}'), V_\tau'(\mathbf{r}'))]. \quad (32)$$

All Lagrangian transformations are reversible, so the reverse of  $T_{i,j}^f(\mathbf{r}, \mathbf{r}')$  takes  $\mathbf{B}_f(\mathbf{r})$  into  $\mathbf{B}_\tau(\mathbf{r})$ . The transformation  $T_{i,j}^\tau(\mathbf{r}, \mathbf{r}')$  is conceptually relevant to the two-plate problem in ensuring  $\mathbf{B}_\tau(\mathbf{r})$  has globally defined flux functions by virtue of  $\mathbf{B}_0$ , but is not directly related to  $\mathbf{B}_f(\mathbf{r})$ .

The Theorem can be seen in terms of the general absence of an analytical solution  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  to the above boundary value problem. Although nonlinear PDEs (24) and (25) are formidable, their basic mathematical features can be understood with the physics of the Theorem treated in the monograph and the subsequent works derived from it. It bears reminding that analyticity is an assumption extraneous to the physics of the problem. When the topology of  $\mathbf{B}_\tau(\mathbf{r})$  is such that the boundary value problem yields no analytical solution, the physics of the problem must give up the assumption of analyticity and allow for  $\mathbf{B}_f(\mathbf{r})$  to possess TDs as a weak solution of the force-free equations.

There is an incomplete topological relationship between  $[U_f, V_f]$  and  $[U_\tau, V_\tau]$ , contained in boundary conditions (29) and (30) that demand for  $\mathbf{B}_f(\mathbf{r})$  and  $\mathbf{B}_\tau(\mathbf{r})$  to possess the same mapping of the foot-points on  $z = 0$  onto  $z = L$ . Now, the preparation of  $\mathbf{B}_\tau(\mathbf{r})$  from  $\mathbf{B}_0$  endows  $\mathbf{B}_\tau(\mathbf{r})$  with two topological properties, the said foot-point mapping as well as a specific interweaving of the field lines going from boundary to boundary. The mathematical structure of the boundary-value problem does not permit both the foot-point mapping and the field-line interweaving to be imposed on the field solution  $\mathbf{B}_f(\mathbf{r})$ .

Subject to imposing only the foot-point mapping, an analytical  $\mathbf{B}_f(\mathbf{r})$ , assuming it exists as determined by the boundary value problem, may or may not be topologically equivalent to the given  $\mathbf{B}_\tau(\mathbf{r})$ . If the two fields are topologically equivalent, we would, indeed, have found the proper solution  $\mathbf{B}_f(\mathbf{r})$  for the given  $\mathbf{B}_\tau(\mathbf{r})$ . Their topological equivalence is then described by a specific continuous fluid displacement  $T_{i,j}^f(\mathbf{r}, \mathbf{r}')$  that relates their respective

pairs of flux functions stipulated by equation (32).

There are two other possibilities to consider, one being the case of the existing analytical solution  $\mathbf{B}_f(\mathbf{r})$  found not to be topologically equivalent to  $\mathbf{B}_\tau(\mathbf{r})$ , that is,  $\mathbf{B}_f(\mathbf{r}) \notin \mathcal{B}(\tau)$ . The other case is that the boundary-value problem has no solution. In either cases the solution to the physical problem is then a field  $\mathbf{B}_f(\mathbf{r})$  embedding equilibrium TDs, whose flux functions  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  are weak solutions. Then, there is no simple continuous displacement transformation  $T_{i,j}^f(\mathbf{r}, \mathbf{r}')$  to speak of. Instead we must deal with an infinite sequence of fluid displacements  $T_{i,j}^{\{\tau \rightarrow f\}}(\mathbf{r}, \mathbf{r}')$  that takes  $[(U'_\tau(\mathbf{r}'), V'_\tau(\mathbf{r}'))]$  to as close to the weak solution  $[U_f(\mathbf{r}), V_f(\mathbf{r})]$  as desired.

### C. A perspective view of TD formation

The frozen-in interaction among three flux systems is basic to TD formation, depicted as a 3D process in Figure 1c. The magenta and white flux surfaces belong to two local flux systems that converge to make TD contact through a hole or gap in the green flux surface belonging to a third intervening flux system that has been locally expelled out of the way. Actually, similar holes in a laminated layer of flux surfaces of the third system create a tunnel through which the two converging systems enter to meet each other. A contorted flux surface keeps the three systems topologically invariant and distinct. This and other displaced flux surfaces become a complex of TDs during a dynamical evolution towards a minimum-energy state. Each TD attains force balance by equalizing the magnetic pressures on its two sides as the current density aligns with the continuous part of the field. Here we present four instructive demonstrations of TD formation to complete our analysis of the two-plate problem.

#### 1. *The force-free field as a small-parameter power series*

The Theorem was discovered in the Parker study of a power-series solution<sup>1</sup> for  $\mathbf{B}_f(\mathbf{r})$  to the two-plate problem,

$$\begin{aligned} \mathbf{B}_f &= \mathbf{B}_0 + \mathbf{b} \\ &\equiv B_0 \hat{\mathbf{z}} + \sum_{n=1}^{\infty} \mathbf{b}_n \epsilon^n, \quad \epsilon \ll 1, \\ z = 0, L, \quad b_{n,z} &= 0 \quad \text{for all } n. \end{aligned} \tag{33}$$

Let us assume analyticity by which  $\nabla \times (\nabla p) \equiv 0$  for any scalar function  $p(\mathbf{r})$ . Therefore, force-free equation (14) imposes the necessary condition on the tension force,

$$\nabla \times [(\mathbf{B} \cdot \nabla) \mathbf{B}] = 0, \quad (34)$$

for the field to be analytical. In the two-plate problem, the power-series force-free field  $\mathbf{B}_f(\mathbf{r})$  is to be topologically equivalent to  $\mathbf{B}_\tau(\mathbf{r})$  similarly expressed as a power series. Without needing to deal with this formidable requirement, Parker first proved by mathematical induction that the necessary condition (34) implies

$$\frac{\partial}{\partial z} \mathbf{b}_n = 0, \quad \text{for all } n, \quad \text{in the limit } \epsilon \rightarrow 0. \quad (35)$$

That is, the force-free field  $\mathbf{B}_f(\mathbf{r})$  can have no  $z$  variation in the limit  $\epsilon \rightarrow 0$ .

This rigorous result arises physically from the anisotropic nature of the Lorentz force. The tension force at each point in space lies in the plane of the tangent and normal to the field line at the point. Whereas, the pressure force acts along the normal to the surface of constant field pressure at the point. Equation (34) is a global condition on the geometric compatibility for the two forces to balance everywhere in space, a compatibility not always possible under the assumption of analyticity.

Let  $\epsilon$  characterize the size of the horizontal spatial variations of the twisted initial field  $\mathbf{B}_\tau(\mathbf{r})$  in Figure 1b. Under the assumption of analyticity, the thicknesses of the boundary layers at  $z = 0, L$  may be expected to be of the order of  $\epsilon$  and one might neglect them in the limit  $\epsilon \rightarrow 0$ . Although the result (35) demanding  $\frac{d}{dz} \mathbf{B}_f(\mathbf{r}) = 0$  is obtained in that limit, it reveals that analyticity is generally overly restrictive for a force-free field. The initial field  $\mathbf{B}_\tau(\mathbf{r})$  is arbitrarily prescribed, generally possessing an irremovable  $z$ -dependence in its field-line interweaving that is inherited by  $\mathbf{B}_f(\mathbf{r})$ . For field topologies with irremovable  $z$  dependence, their force-free states must embed equilibrium TDs.

Two noteworthy corollaries follow. Firstly, TDs forming in  $\mathbf{B}_f(\mathbf{r})$  must extend along flux surfaces to intersect  $z = 0, L$  where the the field intensities may be unbounded while the TD discrete currents remain bounded, a remarkable demonstrable property of boundary rigidity<sup>19,20</sup>. In other words, the boundary layers at  $z = 0, L$  are mathematically singular in a complex manner. Secondly, **the** overly restrictive condition  $\frac{d}{dz} \mathbf{B}_f(\mathbf{r}) = 0$  imposed by analyticity implies that the analytical force-free fields are a continuum that is conceptually of *a far smaller size* than the continuum of TD-embedding force-free fields.

## 2. *Multiple-flux interaction*

Consider the Lundquist force-free field,

$$\mathbf{B} = J_1(\alpha_0 R)\hat{\varphi} + J_0(\alpha_0 R)\hat{\mathbf{z}} \quad (36)$$

in cylindrical coordinates  $(R, \varphi, z)$ , a solution to force-free equations (11) and (12) with  $\alpha = \alpha_0$ , a positive constant. Replacing the  $z$  component with  $B_0$ , a constant, we obtain a non-equilibrium cylindrical field confined within a finite radius

$$\mathbf{B}_{\alpha_0} = J_1(\alpha_0 R)\hat{\varphi} + B_0\hat{\mathbf{z}}, \quad R \leq R_0, \quad (37)$$

where  $R_0$  is related to the constant  $\alpha_0$  given by the first zero of the Bessel function  $J_1(\alpha_0 R_0) = 2.83$ ; see Figure 2a. A rich variety of the initial, global non-equilibrium field  $\mathbf{B}_\tau(\mathbf{r})$  for the two-plate problem can be constructed out of cylindrical field (37). Each global field varies only with  $(x, y)$  and is composed of the uniform field  $\mathbf{B}_0$  studded with any chosen number of the cylindrical field of an  $\alpha_0$ -value that occupy non-overlapping cylindrical volumes. Notably, two or more of these cylindrical fields of different cylindrical sizes can be juxtaposed to be in tangential contact.

Figure 2b shows the simplest case of a periodic 2D array of identical non-equilibrium cylindrical fields, taken from the monograph. With sufficiently strong twist built into them, the cylindrical fields would all expand to create gaps in the exterior initially uniform field. In the relaxed force-free state, the expanded cylindrical fields are hexagonally shaped, displacing and compressing their untwisted external field into unconnected interstitial spaces. The flux surfaces that keep the cylindrical fields and their external fields topologically distinct, naturally become TDs, subject to the continuity of the magnetic pressure across them. If  $L$  is sufficiently larger than  $R_0$ , the field away from the boundaries  $z = 0, L$  would yield to the expansion of the cylindrical fields as described. Whereas, field displacements are greatly restrained in the two boundary layers along these rigid boundaries. These TDs intersect the anchoring boundaries where the field exhibits a severe mathematical singularity<sup>19,20</sup>.

TDs can also form inside each cylindrical field, a complicated process possible to infer from the simple case of single cylindrical field studded into  $\mathbf{B}_0$  depicted in Figure 3a. The relaxation of this field proceeds with an initially non-zero Lorentz force in  $R < R_0$ ,

$$\begin{aligned} (\nabla \times \mathbf{B}_{\alpha_0}) \times \mathbf{B}_{\alpha_0} &= - \left[ \frac{1}{2} \frac{d}{dR} (B_\varphi^2) + \frac{B_\varphi^2}{R} \right] \hat{R} \\ &= -J_1(\alpha_0 R)J_0(\alpha_0 R)\hat{R} \end{aligned} \quad (38)$$

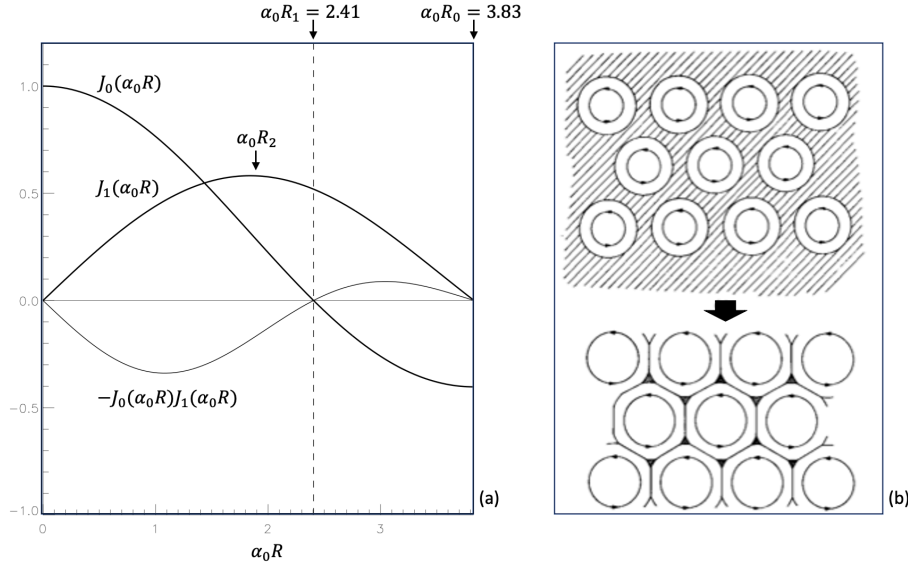


FIG. 2. (a) The Lundquist  $B_\varphi(R) = J_1(\alpha_0 R)$ . (b) Sectional views of a periodic array of identical, right-handedly twisted, cylindrical fields studded into a (hatched) uniform field  $\mathbf{B}_0$ , not in equilibrium (top) and in a relaxed force-free state (bottom), the latter showing the expanded cylindrical fields in hexagonal TD-contact and the compressed, external untwisted field trapped in unconnected interstitial spaces.

as the sum of a pressure force and a tension force defined by  $B_\varphi$ . The tension force  $\frac{B_\varphi^2}{R}$  acts radially inward everywhere. The pressure  $B_\varphi^2$  vanishes at  $R = 0, R_0$  and is maximum at  $R = R_2$ ; see Figure 2a, The Lorentz force is thus inward in  $0 < R < R_1$  and outward in  $R_1 < R < R_0$ ,  $R_1$  defined by the first zero of Bessel function  $J_0(\alpha_0 R_1) = 0$  with  $R_2 < R_1$ . The non-equilibrium cylindrical field would thus expand in opposite radial directions from  $R = R_1$ , the radially inward expansion driving a classical pinch and the radially outward expansion pushing into the external, initially uniform field. The rigid anchoring of the field at  $z = 0, L$  immediately results in a significant dependence on  $z$  in the relaxed field. Around the  $R = 0$  axis, the magnetic twist in the narrowed, magnetic flux tube propagates in opposite directions to the boundaries to be accumulated in the funnel-shaped, anchored parts of the flux tube<sup>21,22</sup>. In contrast, in the outer cylindrical shell of magnetic flux, away from the  $R = 0$  axis, magnetic twist propagates away from the boundaries to accumulate about the mid-plane  $z = L/2$  where the flux is expanded, sketched in Figure 3a.

This is an intriguing theoretical phenomenon worthy of further investigation. Figure

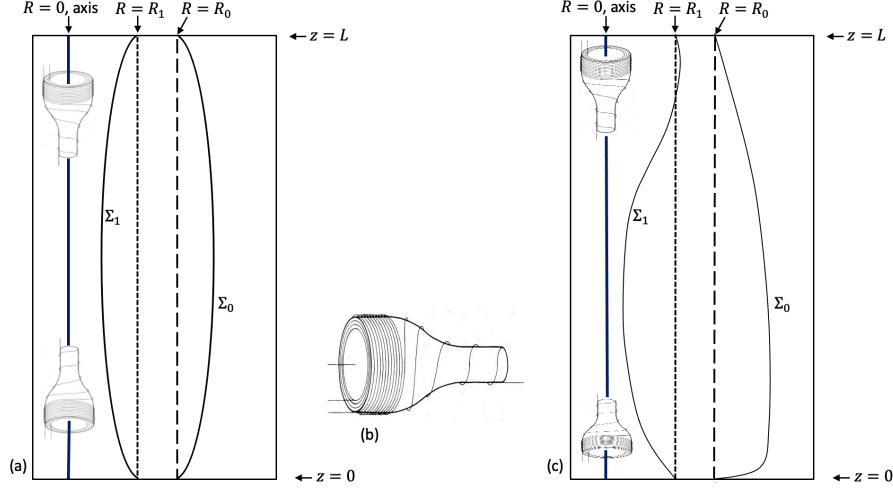


FIG. 3. (a) The domain  $0 < R$ ,  $0 < z < L$  of a single non-equilibrium cylindrical field initially continued at  $R = R_0$  into the external uniform field  $\mathbf{B}_0$  described in text. Relaxation to a force-free state involves the flux surfaces  $\Sigma_0$  and  $\Sigma_1$  deforming, as sketched, from their initial cylindrical shapes (dashed lines) while anchored to  $z = 0, L$  at  $R = R_0$  and  $R = R_1$ , respectively. Symmetry about the mid-plane  $z = 0.5L$  is assumed. (b) A 3D sketch from<sup>21</sup> showing the dynamical accumulation of magnetic twists at the widened end of a cylindrical flux surface that has a gradual spatial increase of its cylindrical radius along the length of the cylinder. This sketch is reproduced in sub-figures (a) and (c), appropriately sized and oriented to indicate magnetic twist transport in the pinched field around the axis  $R = 0$ . (c) Consequence of breaking the mid-plane symmetry assumed in the dynamical development in (a). The detail dynamics of the force-free relaxation would determine a critical, intervening flux surface (not shown) between  $\Sigma_0$  and  $\Sigma_1$ , that separates the pinched axial flux from its shell of outer flux, magnetic twist accumulating toward  $z = 0, L$  in the former versus accumulating away from  $z = 0, L$  in the latter flux. This critical flux surface as well as  $\Sigma_0$  may be inferred to be among the TDs forming during the relaxation.

3a assumes symmetry about the mid-plane. Breaking this symmetry would allow unequal discrete amounts of twist accumulating to the boundaries in the pinched flux around the  $R = 0$  axis, sketched in Figure 3c. Similarly, unequal discrete amounts of twist would propagate from the boundaries to accumulate about the mid-plane where the outer shell of flux has expanded. The critical flux surface separating these two dynamically distinct flux regions would then become a TD<sup>23</sup>.



### 3. *The Parker optical analog*

Gaps in the flux surfaces of a force-free field have an optical analogy<sup>24</sup>. The respective flux surfaces of a force-free field  $\mathbf{B}$  and its field-aligned current density coincide. Therefore, the curl of  $\mathbf{B}$  has no component perpendicular to a flux surface, and  $\mathbf{B}$  must be a potential field in the flux surface,

$$\begin{aligned}\mathbf{B}_{\text{ff}} &= \nabla\phi(\xi_1, \xi_2), \\ &= \frac{1}{h_1} \frac{\partial\phi}{\partial\xi_1} \hat{\xi}_1 + \frac{1}{h_2} \frac{\partial\phi}{\partial\xi_2} \hat{\xi}_2,\end{aligned}\tag{39}$$

described by orthogonal coordinates  $(\xi_1, \xi_2)$  with the Lamé coefficients  $[h_1(\xi_1, \xi_2), h_2(\xi_1, \xi_2)]$  that define the gradient operator and the path-length element  $ds^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2$ . On this flux surface, the field lines are described by the differential equations

$$\begin{aligned}Bh_1 \frac{d\xi_1}{ds} &= \frac{1}{h_1} \frac{\partial\phi}{\partial\xi_1}, \\ Bh_2 \frac{d\xi_2}{ds} &= \frac{1}{h_2} \frac{\partial\phi}{\partial\xi_2},\end{aligned}\tag{40}$$

where  $B(\xi_1, \xi_2)$  is the field amplitude, whereas  $B^2$  and the potential  $\phi$  are related by the first order PDE,

$$[\nabla\phi(\xi_1, \xi_2)]^2 = \left(\frac{1}{h_1} \frac{\partial\phi}{\partial\xi_1}\right)^2 + \left(\frac{1}{h_2} \frac{\partial\phi}{\partial\xi_2}\right)^2 = B^2,\tag{41}$$

recognizable as the equation for optical ray paths in a medium with  $\phi$  and  $B$  as the optical eikonal and refractive index, respectively. Thus, differential equations (40) describe the field lines as optical rays streaming and refracted by a variable refractive index  $B(\xi_1, \xi_2)$ .

The bending of light-rays around a region of high refractive index (i.e., strong amplitude  $B$ ) has a concavity of the appropriate sense to correspond to the tension force balancing the outward pressure force. This lensing effect can be so severe that the rays are optically excluded from a local region containing a maximum refractive index. In analogy, the balance between the magnetic tension and pressure forces may require the displacement of field lines to open up gaps in a flux surface where none existed in the continuous field not in equilibrium. Through those gaps, magnetic fluxes external to the flux surface meet one another to form TDs.

Note that TD equilibrium requires  $B^2$  to be continuous, i.e., these gaps involve no discontinuity in refractive index  $B$  in the optical analog. Whereas, the vector field  $\mathbf{B}$  is discontinuous across TD. Not only has analyticity broken down, gaps in flux surfaces must appear

in the force-free field  $\mathbf{B}_f(\mathbf{r})$  where none existed in the initial field  $\mathbf{B}_\tau$ , two reasons for the  $(U, V)$  boundary value problem to possess no analytical solution.

#### 4. *Multiple flux interaction in the Cartesian plane*

Spontaneous current sheets take a simple form for three or more planar flux systems interacting in the unbounded  $y - z$  space above boundary  $z = 0$  where magnetic foot-points are rigidly anchored. The properties discussed below are well known<sup>2,15,19,25</sup> that now take an instructive significance when examined in relation to the Theorem as developed in Subsections A and B.

With  $x$  as an ignorable coordinate, consider the solenoidal field lying in the  $y - z$  plane with its current density,

$$\mathbf{B} = \nabla x \times \nabla A(y, z), \quad (42)$$

$$\nabla \times \mathbf{B} = -\nabla^2 A \hat{\mathbf{x}}, \quad (43)$$

in terms of a flux function  $A(y, z)$ . Field lines are curves of constant- $A$  in the  $y - z$  plane. Independent of any topological consideration, the above current density being everywhere perpendicular to its field implies that the field, if force-free, must be potential, satisfying Laplace equation

$$\nabla^2 A = 0 \quad (44)$$

subject to a given distribution of  $A(y, 0)$  which fixes the normal field component  $B_z$  on  $z = 0$ . We also impose the condition that  $\mathbf{B} \cdot \mathbf{B}$  vanishes in the far  $\sqrt{y^2 + z^2} \rightarrow \infty$  to treat fields with finite energy in  $z > 0$ . Let  $\mathbf{B}_{pot} = \hat{\mathbf{x}} \times \nabla A_{pot}(y, z)$  denote the unique potential field fixed by  $A(y, 0)$ .

Figure 4a is a sketch of a three-flux field that is symmetric in  $y$  and not force-free in  $z > 0$ , with a specific  $A(y, 0)$  giving a boundary  $B_z(y, 0)$  that reverses sign on three locations on  $z = 0$ . The field has a specific topology  $\tau$  defined by the map of boundary  $z = 0$  onto itself from one boundary foot-point to the other of each field line as a curve of constant- $A$ . This field is one of the continuum of continuous fields of all topologies admissible in  $z > 0$  for a fixed  $A(y, 0)$ . In the language of Subsection A, the vector-function space of all these continuous fields may be decomposed into the union of the disjoint sub-spaces  $\mathcal{B}(\tau)$  each

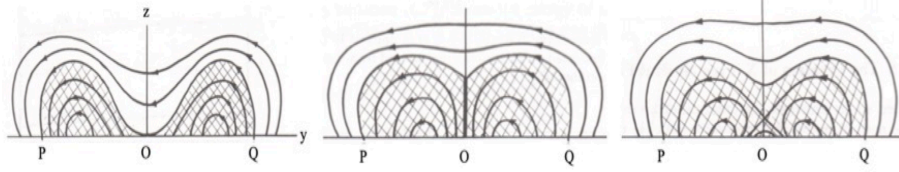


FIG. 4. Three fields admitted by a fixed boundary  $A(y, 0)$  distribution on  $z = 0$ . (a) A non-equilibrium field comprising 3 flux systems with no neutral point. (b) Equilibrium field embedding a vertical equilibrium TD and topologically equivalent to field (a). (c) The potential field comprising 4 flux systems with an X-type neutral point, not topologically equivalent to fields (a) and (b); no other analytical force-free field than field (c) exists for the fixed  $A(y, 0)$ .

containing fields of the same topology  $\tau$ ,

$$\mathcal{B}_{all} = \bigcup_{\tau} \mathcal{B}(\tau). \quad (45)$$

Under the frozen-in condition, with the foot-points anchored rigidly on  $z = 0$ , any field is continuously deformable only into another of the same topology  $\tau$ .

The force-free field  $\mathbf{B}_f$  of a chosen  $\tau$  is given by the modified Dirichlet variational principle,

$$\delta \mathcal{E} = \delta \int_{z>0} \mathbf{B}_\tau \cdot \mathbf{B}_\tau \, dydz \equiv \delta \int_{z>0} |\nabla A|^2 \, dydz = 0, \quad \mathbf{B}_\tau \in \mathcal{B}(\tau), \quad (46)$$

that is, the variational extremum is taken over the subspace  $\mathcal{B}(\tau)$  instead of the space  $\mathcal{B}_{all}$  of all fields<sup>15</sup>. The given  $A(y, 0)$  alone determines a unique potential field  $\mathbf{B}_{pot}$ , the only analytical force-free field admissible by  $A(y, 0)$ . Therefore, the extremum problem (46) has no analytical solution for  $A$  for all chosen topologies except for the one  $\tau = \tau_{pot}$  of  $\mathbf{B}_{pot}$ . For  $\tau \neq \tau_{pot}$ , the extremum problem has a physical, non-analytical solution  $A$  with discontinuous first derivatives, describing a TD bearing force-free field.

Figure 4(c) is a sketch of the case of a potential field  $\mathbf{B}_{pot}$  determined by  $A(y, 0)$  fixed by the non-force-free field  $\mathbf{B}_{\tau^*}$  in Figure 4(a). The former is a four-flux field in a topology  $\tau_{pot}$  containing an X-type neutral point, whereas the latter is a three-flux field with a specific topology  $\tau^*$  containing no neutral point, the two fields not topologically equivalent. Figure 4(b) is a sketch of the force-free field  $\mathbf{B}_f$  embedding a TD, given by the weak solution  $A_f$  to the extremum problem (46) for  $\tau = \tau^*$ . Direct calculation shows that  $A_f$  satisfies the Poisson equation

$$\nabla^2 A = f(z)\delta(y), \quad (47)$$

subject to the given  $A(y, 0)$  and the far boundary condition for a finite magnetic energy. Here  $f(z) \neq 0$  in  $0 < z < z_0$ , describing a vertical current sheet extending to a height  $z_0$ , of a profile such that  $\mathbf{B}_f$  is topologically equivalent to  $\mathbf{B}_{\tau^*}$ . Relating to the two-plate initial boundary-value problem, viscous relaxation would take a field in the initial state  $\mathbf{B}_{\tau^*}$  along an evolutionary path in subspace  $\mathcal{B}(\tau^*)$  to  $\mathbf{B}_f$  as the terminal, TD-embedding force-free field.

This example illustrates the general extreme paucity of analytical force-free states in a perfectly conducting fluid. Fixed by  $A(y, 0)$  describing where a field enters and leaves the domain  $z > 0$ , the fields of all topologies, with just one exception, can attain force-free equilibrium only by embedding TDs. This paucity of analytical force-free states parallels the Theorem Parker discovered in his  $\epsilon$ -expansion force-free fields. In the latter, a continuum of analytical force-free fields exists but they are over restricted to be independent of  $z$ . These fields are *sparsely* distributed among the continuum of TD bearing  $z$ -dependent force-free fields.

The solenoidal field with  $x$  as an ignorable coordinate may have three components,

$$\mathbf{B} = B_x(y, z) \hat{\mathbf{x}} + \nabla x \times \nabla A(y, z), \quad (48)$$

$$\nabla \times \mathbf{B} = -\nabla^2 A \hat{\mathbf{x}} - \nabla x \times \nabla B_x(y, z). \quad (49)$$

Force-free equations (11) and (12) impose the condition that  $B_x(y, z)$  varies as a strict function of  $A(y, z)$  and reduce to the Shrafranov equations,

$$\begin{aligned} \nabla^2 A + B_x(A) \frac{dB_x(A)}{dA} &= 0, \\ \alpha &= \frac{dB_x(A)}{dA}. \end{aligned} \quad (50)$$

With  $\alpha \neq 0$ , this more general class of force-free fields admits field aligned current densities. In this class the same basic property of continuous force-free fields being distributed sparsely among the TD force-free fields is found in even richer diversity<sup>19,20</sup>. This general sparsity of continuous equilibrium fields was not appreciated at the time the 2D force-free fields were studied, but an important point was recognized then, that the presence of a magnetic neutral point is not essential for the formation of a TD<sup>25-27</sup>.

### III. OVERVIEW

The Magnetostatic Theorem is not strictly magnetostatic, of course. The general inevitable presence of TDs in an equilibrium field of an arbitrary 3D topology is due to the readiness, under the frozen-in condition, for a flux system to be expelled out of the way and let two other flux systems make direct contact. The formation of a TD is a localized process but the inevitability of this process occurring somewhere or everywhere in the field is a global process driven by the topological invariance of the field. In the two-plate problem, topological invariance takes the simplest form of an unchanging topology in the rigidly anchored field. Thus, the physical and mathematical reasonings of the Theorem could proceed in this problem by using the concept of topological equivalence without having to explicitly deal with the field's interesting but formidable topological properties<sup>4,28-35</sup>. That is, the rigid boundary conditions have no significance beyond being a theoretical simplification for a straightforward demonstration of TD formation as a universal property of the frozen-in field.

In a compressible, perfectly-conducting hydromagnetic fluid, mathematical discontinuities of two kinds are physically admissible, namely, the hydromagnetic shock waves, associated with the fast and slow mode continuous waves; and the moving contact fluid surfaces. The latter include boundaries separating different kinds of fluids as well as the magnetic rotational discontinuities associated with the Alfvén waves<sup>36</sup>. Thus, the corona evolving quasi-steadily to the slow moving photosphere is naturally pervaded with near-equilibrium TDs forming under the condition  $\mathcal{L}_\eta \gg 1$ , the TDs forming only to dissipate by resistive reconnection that leads to a change in field topology and further TD formation, as we have described.

The continuum of analytical force-free fields being sparsely immersed<sup>37</sup> into the "even larger" continuum of TD-embedded force-free fields has a probabilistic interpretation<sup>6,23</sup>. A random pick from the continuum of *all* the admissible field topologies in a hydromagnetic system has essentially zero chance of being a topology compatible with an analytical force-free field. This corollary of the Theorem is the basis for the assertion that, following the dissipation of a TD under a breakdown of the condition  $\mathcal{L}_\eta \gg 1$ , the likelihood for spontaneous TDs returns with the restoration of the high-conductivity condition. This is the attractive feature of the Parker theory of coronal heating as a turbulently sustained

hydromagnetic process. By this process, near perfect conductivity is as much a consequence of as it is the origin of the maintained high temperature of the fully ionized corona.

The Theorem is a radical complement to the classical hydromagnetic approach<sup>9</sup> of perturbing a given continuous equilibrium field to discover its instability to linear and nonlinear perturbations. The complementary approach deals with a fundamentally distinct effect, that equilibrium and field analyticity are generally incompatible if field topology is to be preserved under the condition of perfect conductivity. That is, continuous equilibrium fields may be in the probabilistic sense non-existent under general astrophysical circumstances. It is intriguingly instructive to think of a circumstance under which a system admits only one continuous equilibrium state that is unstable<sup>38</sup>. Perturbing this field must result in an evolution towards a lower-energy equilibrium that must be a field embedding one or more TDs. The Theorem is at the root of many studies on field topology as the global driver of TDs and their subsequent dissipation via magnetic reconnection<sup>39–46</sup>.

The present paper has laid out the elements of the Theorem, viewed from its variational formulation and from the surgically-detailed mathematical nature of the force-free equations in terms of the Euler flux functions. Parker’s concept-transforming analysis in his monograph may be assessed to have given us a physically complete understanding of the Theorem, whereas Section II has given a survey of many unsolved magnetostatic problems worthy of effort to advance the Theorem. The formation of TDs in multiple flux systems in the Cartesian plane, an old result interpreted afresh in light of the Theorem in Section II.C.4, parallels the corresponding TD formation in Parker’s  $\epsilon$ -expansion 3D force-free fields. This interpretation gives a glimpse of the wealth of spontaneous current-sheet processes yet to be discovered.

We have limited our attention to the force-free fields, to keep the physical problem simple. Irrepressible spontaneous current sheets also operate in more complex hydromagnetic systems, e.g., in the presence of fluid pressure, steady fluid flows, and complex thermal-energy transport in the corona<sup>47,48</sup>. To solve the mathematical problems surveyed in Section II beyond the analytical success we happen to have, direct numerical simulation holds the only realistic promise. The physical understanding of the Theorem provides guidance in formulating numerical simulations<sup>11,46,49–51</sup> capable of handling the required, orders of magnitudes, broad range in computational Lundquist number  $\mathcal{L}_\eta$ . That understanding also guides the search for the implications of the Theorem in coronal observations<sup>6,52</sup>.

## IV. ACKNOWLEDGMENT

This work is dedicated to the late Eugene N. Parker. Section 2 is a product of conversations between the author and E. N. P. during the pandemic years 2020-2022. The author thanks Phil Judge for invaluable comments on this work, and Holly Gilbert and Scott McIntosh for hosting and supporting his visit at the National Center for Atmospheric Research.

## REFERENCES

- <sup>1</sup>Parker, E. N., Topological dissipation and the small-scale fields in turbulent gases, *ApJ* 174, 499, 1972.
- <sup>2</sup>Parker, E. N., *Spontaneous Current Sheets in Magnetic Fields* (Oxford Univ. Press) 1994.
- <sup>3</sup>Parker, E. N., Nanoflares and the solar X-ray corona, *ApJ* 330, 474, 1988.
- <sup>4</sup>Janse, Á. M., B. C. Low & E. N. Parker, Topological complexity and tangential discontinuity in magnetic fields, *Phys. Plasmas* 17, 092901, 2010.
- <sup>5</sup>Judge, P. G., K. Reardon, & G. Cauzzi, Evidence for sheet-like elementary structures in the Sun's atmosphere?, *ApJL* 755, L11, 2012.
- <sup>6</sup>Low, B. C., Chapter 6. Coronal magnetism as a universal phenomenon, *The Sun as a Guide to Stellar Physics*, ed. by O. Engvold, J.-C. Vial & A. Skumanich, (Elsevier) 2019.
- <sup>7</sup>Engvold, O., J.-C. Vial & A. Skumanich (eds) *The Sun as a Guide to Stellar Physics* (Elsevier) 2019.
- <sup>8</sup>Cargill, P., H. P. Warren, S. J. Bradshaw, Modelling nanoflares in active regions and implications for coronal heating mechanisms, *Phil. Trans A Math. Phys. Eng. Sci.* 373, 20140260, 2015.
- <sup>9</sup>Kulsrud, R. M., *Plasma Physics for Astrophysics* (Princeton Univ. Press) 2005.
- <sup>10</sup>Gonzalez, W., & E. Parker (eds.), *Magnetic reconnection: Concepts and applications, astrophysics and space science*, Springer, 2016.
- <sup>11</sup>Low, B. C., Newtonian and non-Newtonian magnetic-field relaxations in solar-coronal MHD, *ApJ* 768, id.7, 2013.
- <sup>12</sup>Low, B. C., & Y. Q. Lou, Modeling solar force-free magnetic fields, *ApJ* 352, 343, 1990.

- <sup>13</sup>Low, B. C., & M. A. Berger, A morphological study of helical coronal magnetic structures, *ApJ* 589, 644, 2003.
- <sup>14</sup>Hu, Y. Q., & B. C. Low, The energy of electric current sheets. I. Models of moving magnetic dipoles, *Solar Phys.* 81, 107, 1982.
- <sup>15</sup>Low, B. C., Spontaneous formation of electric current sheets by the expulsion of magnetic flux, *ApJ* 340, 558, 1989.
- <sup>16</sup>Parker, E. N., *Cosmical Magnetic Fields* (Oxford Univ. Press) 1979.
- <sup>17</sup>Courant, R., & D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, p. 173 (Interscience) 1966.
- <sup>18</sup>Courant, R., & D. Hilbert, *Methods of Mathematical Physics*, Vol. 2, p. 635 (Interscience) 1960.
- <sup>19</sup>Low, B. C., & R. Wolfson, Spontaneous formation of current sheets and the origin of solar flares, *ApJ* 324, 574, 1988.
- <sup>20</sup>Low, B. C., Equilibrium and dynamics of coronal magnetic fields, *ARAA* 28, 491 1990.
- <sup>21</sup>Parker, E. N., The dynamical properties of twisted ropes of magnetic field and the vigor of new active regions on the Sun, *ApJ* 191, 245, 1974.
- <sup>22</sup>Parker, E. N., The origin of solar activity, *ARAA* 15, 45, 1977
- <sup>23</sup>Parker, E. N., *Spontaneous current sheets in solar and stellar coronae*, Lecture posthumously presented by B. C. Low, Crafoord Prize Symposium, 2022.
- <sup>24</sup>Parker, E. N., The optical analogy for vector fields, *Phys. Fluids* B 3, 2652, 1991.
- <sup>25</sup>Low, B. C., Electric current sheet formation in a magnetic field induced by continuous magnetic footpoint displacements, *ApJ* 323, 358, 1987.
- <sup>26</sup>Aly, J. J., Evolving magnetostatic equilibria, in *Interstellar magnetic fields: Observations and theory*, A88-35776 14-90, p. 240 (Springer-Verlag) 1987.
- <sup>27</sup>Moffatt, H. K., Geophysical and astrophysical turbulence, *Advances in Turbulence*, ed. G. Comte-Bellot, J. Meathieu, p. 228 (Springer-Verlag) 1987.
- <sup>28</sup>Taylor, J. B., Relaxation and magnetic reconnection in plasmas, *Rev. Mod. Phys.* 58, 741. 1985.
- <sup>29</sup>Berger, M. A., & G. B. Field, The topology of magnetic helicity, *J. Fluid Mech.* 147, 133, 1984.
- <sup>30</sup>Berger, M. A., Rigorous new limits on magnetic helicity dissipation in the corona, *Geophys. Astrophys. Fluid Dyn.* 30, 79, 1984.



- <sup>31</sup>Moffatt, H. K., Magnetostatic equilibria and analogous Euler flow of arbitrarily complex topology. Part 1. Fundamentals, *J. Fluid Mech.* 159, 359, 1985.
- <sup>32</sup>Rosner, R., B. C. Low, K. Tsinganos, & M. A. Berger, On the relationship between the topology of magnetic field lines and flux surfaces, *Geophys. Astrophys. Fluid Dyn.* 48, 251, 1989.
- <sup>33</sup>Zhang, M., N. Flyer and B. C. Low, Magnetic field confinement in the corona: the role of magnetic helicity accumulation, *ApJ* 646, 575, 2006.
- <sup>34</sup>Low, B. C., Absolute magnetic helicity and the cylindrical magnetic field, *Phys. Plasmas* 18, 052901, 2011.
- <sup>35</sup>Low, B. C., & F. Fang, Cylindrical Taylor states conserving total absolute magnetic helicity, *Phys. Plasmas* 21, 092116, 2014.
- <sup>36</sup>Landau, L. D., & E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon Press) 1960.
- <sup>37</sup>Low, B. C., & A.K. Egan, Steady fall of isothermal, resistive-viscous, compressible fluid across magnetic field, *Phys. Plasmas* 21, id062105, 2014.
- <sup>38</sup>Ng, C. S., & A. Bhattacharjee, equilibrium and current sheet formation in line-tied magnetic fields, *Phys. Plasmas* 5, 4028, 1998.
- <sup>39</sup>Yu, G., Hydrostatic equilibrium of hydromagnetic fields, *ApJ* 181, 1003, 1973.
- <sup>40</sup>Zweibel, E., & A. H. Boozer, Evolution of twisted magnetic fields, *ApJ* 295, 642, 1985.
- <sup>41</sup>Hahm, T. S., & R. M. Kulsrud, Forced magnetic reconnection, *Phys. Fluids* 28, 3412, 1985.
- <sup>42</sup>Low, B. C., On the possibility of electric-current sheets in dense formation, *Phys. Plasmas* 14, 122904, 2007.
- <sup>43</sup>Aly, J. J., & T. Amari, Does the Compression or the Expansion of a Simple Topology Potential Magnetic Field Lead to the Development of Current Sheets?, *ApJL* 709, L99, 2010.
- <sup>44</sup>Bhattacharyya, R., B. C. Low & P. K. Smolarkiewicz, On spontaneous formation of current sheets: Untwisted magnetic fields, *Phys. Plasmas* 17, id. 112901, 2010.
- <sup>45</sup>Boozer, A., Formation of current sheets in magnetic reconnection, *Phys. Plasmas* 21, 072907, 2014.
- <sup>46</sup>Kumar, S., et al., Formation of magnetic discontinuities through viscous relaxation *Phys. Plasmas* 21, 052904, 2014.

- <sup>47</sup>Tsinganos, K. C., R. Rosner & J. Distler, On the topological stability of magnetostatic equilibria, *ApJ* 278, 409.
- <sup>48</sup>Low, B. C., W. Liu, T. Berger & R. Casini, The hydromagnetic interior of a solar quiescent prominence. II. Magnetic discontinuities and cross-field mass transport, *ApJ* 757, id. 21, 2012.
- <sup>49</sup>Rappazzo, A. F., & E. N. Parker, Current sheets formation in tangled coronal magnetic fields, *ApJ* 773, id. L2, 2013.
- <sup>50</sup>Manchester IV, W., T. Gombosi, F. Y. DeZeeuw, Eruption of a buoyantly emergent magnetic flux rope, *ApJ* 610, 588, 2004.
- <sup>51</sup>Stone, J. M., The ZEUS code for astrophysical magnetohydrodynamics: new extensions and applications, *J. Comp. Applied Math.* 109, 261, 1999.
- <sup>52</sup>Low, B. C., Corona mass ejections, magnetic flux rope, and solar magnetism, *J. Geophys. R.* 106, 25141, 2001.