SO FAR:

- $V = IR$
- KCL, KVL
- SERIES & PARALLEL
- NODE VOLTAGE ANALYSIS
- THEVENIN & NORTON EQUIVALENTS

- BJTs, DIODES
- OP AMPS
- AMPLIFIERS & ANALOG COMPUTERS
- COMPARATORS & SCHMITT TRIGGERS

TODAY:

CH. 51, COMPLEX NUMBERS (USED THROUGHOUT PHYSICS!)

TODAY'S LECTURE PREPARES US TO DEAL WITH CAPACITORS AND INDUCTORS, USING COMPLEX IMPEDANCE.

COMING UP:

- ALTERNATING CURRENT (CH. 7)
- CAPACITORS, INDUCTORS, IMPEDANCE (CH. 8)
- AC CIRCUIT ANALYSIS (CH. 9-10)
- TRANSIENT RESPONSE (CH. 12-13)

COMPLEX NUMBERS (CH. 51)

$$j \equiv \sqrt{-1}$$

MATHEMATICIANS AND PHYSICISTS NORMALLY USE $i = \sqrt{-1}$, WHILE EE'S USE $j = \sqrt{-1}$. SINCE OUR CURRENTS ARE i, WE WILL FOLLOW THE EE CONVENTION.

EXAMPLE:

$$x^2 + 1 = 0$$

SOLVE FOR x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0-4}}{2} = \pm\sqrt{-1} = \pm j$$

EXAMPLE:

$$2x^2 - 2x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 1}}{4} = \frac{+2 \pm 2\sqrt{-1}}{4}$$

$$= \frac{1}{2} \pm \frac{1}{2} j$$

REAL NUMBERS, $\mathbb{R}$ — WHAT DOES $x \in \mathbb{R}$ MEAN?

IMAGINARY NUMBERS, $\mathbb{I}$

CAN BE EXPRESSED AS $z = jy$, $y \in \mathbb{R}$

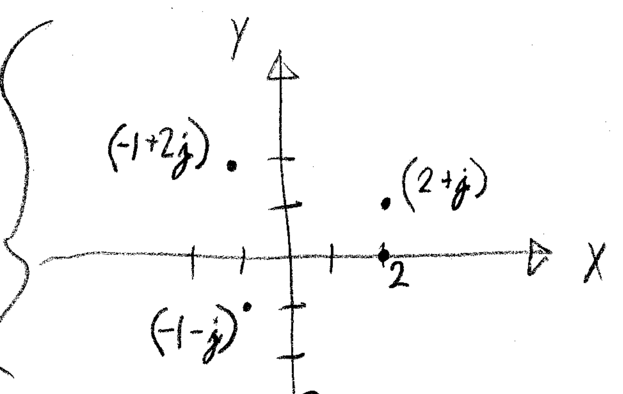
THE IMAGINARY NUMBERS ARE NOT CLOSED UNDER MULTIPLICATION:

$$z^2 = (jy)^2 = -y^2 \in \mathbb{R}$$

COMPLEX NUMBERS, $\mathbb{C}$

CAN BE EXPRESSED $z = x + jy$; $x, y \in \mathbb{R}$

COMPLEX #s CAN BE PLOTTED IN THE COMPLEX PLANE



REAL AND IMAGINARY PARTS:

LET $z = x + jy$.

$\mathrm{Re}(z) \equiv x$

$\mathrm{Im}(z) \equiv y$

NOTE $\mathrm{Im}(z) \in \mathbb{R}$.

☆ ALL QUADRATICS HAVE 2 COMPLEX ROOTS.

COMPLEX CONJUGATE (*):

LET $z = x + jy$. THE COMPLEX CONJUGATE OF z IS DEFINED:

$$z^* \equiv x - jy.$$

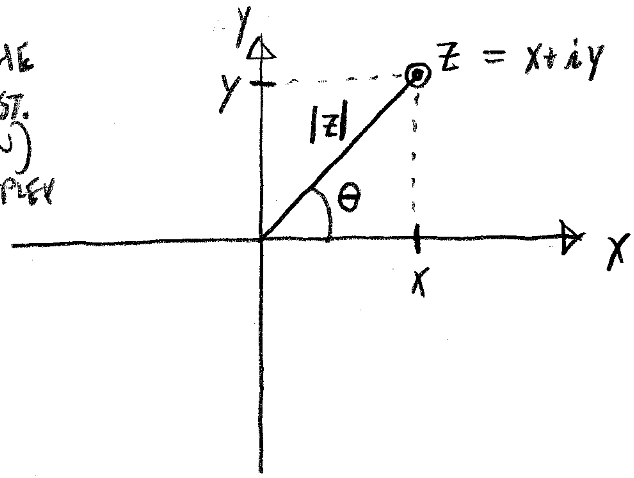
THE ABSOLUTE SQUARE OF z IS DEFINED:

$$|z|^2 = z^* z = (x - jy)(x + jy) = x^2 + y^2$$

THE MAGNITUDE OF z IS DEFINED:

$$|z| = \sqrt{|z|^2} = \sqrt{x^2 + y^2}$$

$|z|$ IS THE RADIUS (DIST. FROM ORIGIN) IN THE COMPLEX PLANE.



THE "PHASE ANGLE" θ IS RELATED TO x & y:

$$\tan \theta = \frac{y}{x}$$

LET $a \in \mathbb{R}$. IT IS EASY TO SHOW THAT:

$$|az| = |a||z|$$

TO SIMPLIFY A RATIO OF COMPLEX NUMBERS, MULTIPLY THE DENOMINATOR BY ITS OWN COMPLEX CONJUGATE:

$$\frac{a + jb}{c + jd} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + j\,\frac{bc - ad}{c^2 + d^2}$$

THE EXPONENTIAL FUNCTION CAN BE DEFINED BY ITS POWER SERIES:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

WHAT HAPPENS WHEN WE GIVE IT AN <u>IMAGINARY</u> <u>ARGUMENT</u>,

$jy \in \mathbb{I} \quad (y \in \mathbb{R})$?

$$e^{jy} = 1 + jy - \frac{y^2}{2!} - j\frac{y^3}{3!} + \frac{y^4}{4!} + j\frac{y^5}{5!}$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + j\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)$$

$$\boxed{e^{jy} = \cos y + j \sin y}$$

THIS IS <u>EULER'S</u> <u>FORMULA</u> * $\quad e^{jy} \in \mathbb{C}$.

y IS IN <u>RADIANS</u>.

<u>EXAMPLE</u>

WRITE THE REAL AND IMAGINARY PARTS OF $\exp(x+jy)$.

$$e^{x+jy} = e^x e^{jy}$$

$$= e^x (\cos y + j \sin y)$$

$$\mathrm{Re}\left(e^{x+jy}\right) = e^x \cos y$$

$$\mathrm{Im}\left(e^{x+jy}\right) = e^x \sin y$$

THE <u>MAGNITUDE</u> OF e^{jy} IS <u>UNITY</u>:

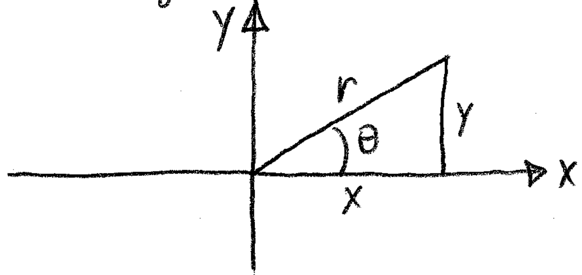
$$|e^{jy}| = \cos^2 y + \sin^2 y = 1$$

ALSO, $|e^{x+jy}| = e^x |e^{jy}| = e^x$.

* SOMETIMES INCORRECTLY CALLED EULER'S THEOREM, WHICH IS <u>DIFFERENT</u>.

SUPPOSE WE WANT TO REPRESENT A COMPLEX NUMBER IN <u>POLAR COORDINATES</u>:

$$z = x + jy$$



$$x = r\cos\theta, \qquad y = r\sin\theta$$

SO $z = r\cos\theta + j r\sin\theta$

$= r(\cos\theta + j\sin\theta)$

$$z = re^{j\theta}$$

INTERPRETATION: $r = |z|$, EUCLIDEAN DISTANCE FROM THE ORIGIN

$e^{j\theta} =$ UNIT VECTOR AT ANGLE θ.

WE CAN CONVERT FROM POLAR TO RECTANGULAR COORDINATES USING

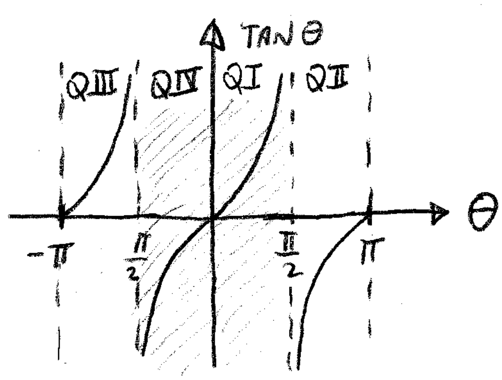
$$\theta = \tan^{-1}\left(\frac{y}{x}\right),$$

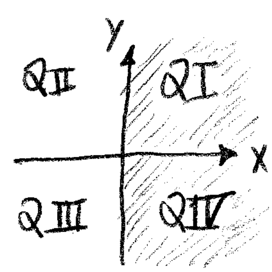
BUT THERE'S A 180° (π RADIANS) AMBIGUITY:

$$\frac{-y}{-x} = \frac{y}{x}, \quad \text{SO} \quad \tan^{-1}\left(\frac{-y}{-x}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

I.E. $\boxed{\tan\theta = \tan(\theta \pm \pi)}$

SIMPLE ATAN() FUNCTIONS GIVE ANSWERS IN THE SHADED REGION, QI/QIV, $-\frac{\pi}{2} < \theta \leq \pi/2$





WORK AROUND THE AMBIGUITY BY NOTICING WHICH QUADRANT $x + jy$ IS IN.

SUPPOSE WE WISH TO SIMPLIFY THE EXPRESSION:

$$z = \frac{\sqrt{2}e^{j\pi/4}}{-1+\sqrt{3}j}$$

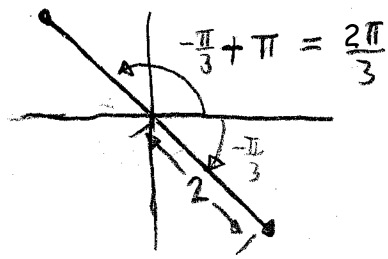
I. IN POLAR COORDINATES:

$$-1+\sqrt{3}j = re^{j\theta}, \quad r = \sqrt{3+1} = 2$$

$$\theta = \tan^{-1}\left(-\sqrt{3}\right) = -1.047 = \frac{-\pi}{3} \quad (-60^\circ)$$

Q: IS θ CORRECT?

A: NO:



SINCE $x<0$ AND $y>0$, WE WANT QII RATHER THAN QIV.

SO $\theta = \frac{2\pi}{3}$.

$$z = \frac{\sqrt{2}\,e^{j\pi/4}}{2\,e^{j2\pi/3}} = \frac{1}{\sqrt{2}}e^{-j\left(\frac{\pi}{4}-\frac{2\pi}{3}\right)} = \frac{1}{\sqrt{2}}e^{-j5\pi/12}$$

II. IN RECTANGULAR COORDINATES:

$$\sqrt{2}\,e^{j\pi/4} = \sqrt{2}\cos\frac{\pi}{4} + j\sqrt{2}\sin\frac{\pi}{4} = 1+j$$

$$z = \frac{(1+j)(-1-\sqrt{3}j)}{(-1+\sqrt{3}j)(-1-\sqrt{3}j)} = \frac{(-1+\sqrt{3})-(1+\sqrt{3})j}{4}$$

$$z = \frac{-1+\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}j$$

TREAT j AS YOU WOULD ANY OTHER ALGEBRAIC SYMBOL. IT OBEYS ALL THE USUAL RULES.