OUTLINE

LAST TIME: (CH. 5)

- COMPLEX NUMBERS
- MAGNITUDE
- COMPLEX CONJUGATE
- EULER'S FORMULA

THIS TIME: (CH. 7)

ALTERNATING CURRENT

- SINE & COSINE, ω, f, T
- POWER
- PHASE SHIFT
- COMPLEX REPRESENTATION (PHASORS)

NEXT TIME: (CH. 8)

- CAPACITORS
- INDUCTORS
- COMPLEX IMPEDANCE

Before we get started on today's topic, Faissler provides the following trig identities (p. 59):

$$\sin^2 a + \cos^2 a = 1 \tag{7-15}$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{7-16}$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{7-17}$$
$$\sin 2a = 2 \sin a \cos a \tag{7-18}$$
$$\cos 2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \tag{7-19}$$

Faissler has some $\pm$ operators in his trig identities. These are not necessary if you consider a & b to be signed quantities. For example, (7-17) implies that

$$\cos(a-b) = \cos a \cos b + \sin a \sin b.$$

You will derive (7-16) and (7-17) for homework. Let's derive (7-18) and (7-19) as an example. Using Euler's formula,

$$e^{2ja} = \cos 2a + j \sin 2a.$$

Alternatively, we could write

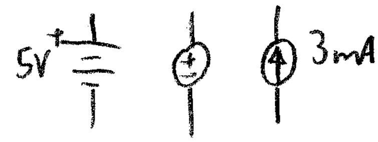
$$e^{2ja} = \left(e^{ja}\right)^2 = (\cos a + j\sin a)^2 = \cos^2 a - \sin^2 a + 2j \cos a \sin a.$$

Either way we go about expressing e^{2ja}, we get a complex result of course. The only way for the two results to match is if both the real *and* the imaginary parts match. That's how we get to (7-18) and (7-19). The later requires also employing (7-15), so that we can have our result in terms of just the cosine, or just the sine.

SO FAR, WE HAVE BEEN DEALING WITH "DIRECT CURRENT" (DC):

$V =$ CONSTANT

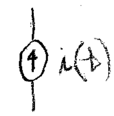
$i =$ CONSTANT

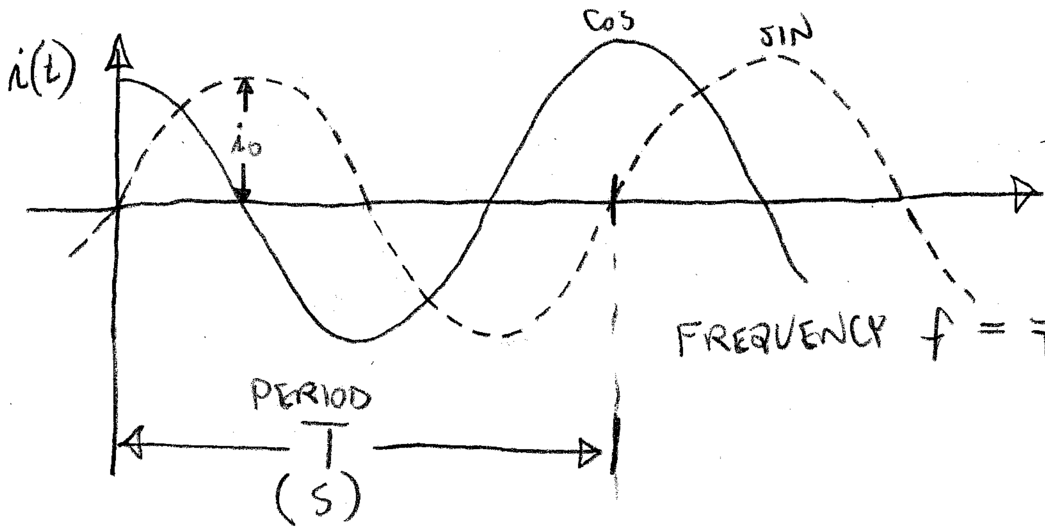


NOW CONSIDER "ALTERNATING CURRENT" (AC). EXAMPLES:

$$i(t) = i_0 \cos(\omega t)$$

OR $$i(t) = i_0 \sin(\omega t)$$





FREQUENCY $f = \frac{1}{T}$ (CYCLES/S HERTZ)

THE QUANTITY ω (RADIANS/SEC) IS CALLED THE "ANGULAR FREQUENCY". IT IS RELATED TO T & f AS FOLLOWS:

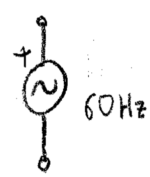
$$\omega T = 2\pi$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega}}$$ SECONDS, S

$$\Rightarrow \boxed{f = \frac{\omega}{2\pi}}$$ HERTZ, Hz

WE MAY SIMILARLY HAVE

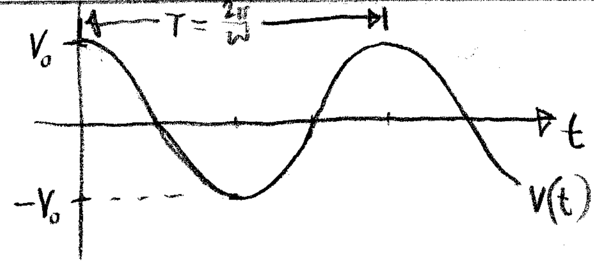
$$V(t) = V_0 \cos(\omega t).$$

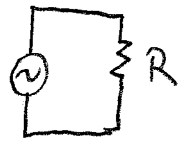


THIS IS STILL CALLED AC (ALTERNATING CURRENT), EVEN THOUGH IT IS VOLTAGE!

SUPPOSE

$$V = V_0 \cos(\omega t)$$



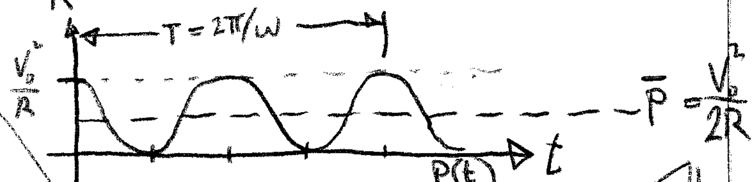


WHAT IS THE POWER DISSIPATED IN THE RESISTOR?

RECALL

$$P = iV \; ; \quad i = \frac{V}{R} = \frac{V_0}{R}\cos(\omega t)$$

$$\therefore \; P = \frac{V_0^2}{R}\cos^2(\omega t) = \frac{V_0^2}{R}\,\frac{1}{2}\left(1 + \cos(2\omega t)\right)$$



WHAT IS THE TIME AVERAGE OF THE POWER?

$$\bar{P} = \frac{1}{T}\int_0^T P\,dt = \frac{1}{T}\int_0^T \frac{V_0^2}{2R}\left(1 + \cos(2\omega t)\right) dt$$

LET $\omega t = \theta$, $dt = \frac{d\theta}{\omega}$

$$\bar{P} = \frac{1}{T}\int_0^{2\pi} \frac{V_0^2}{2R\omega}\left(1 + \cos 2\theta\right) d\theta = \frac{V_0^2}{2R\omega T}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{V_0^2}{2R\omega\left(\frac{2\pi}{\omega}\right)}\,2\pi = \frac{1}{2}\frac{V_0^2}{R}$$

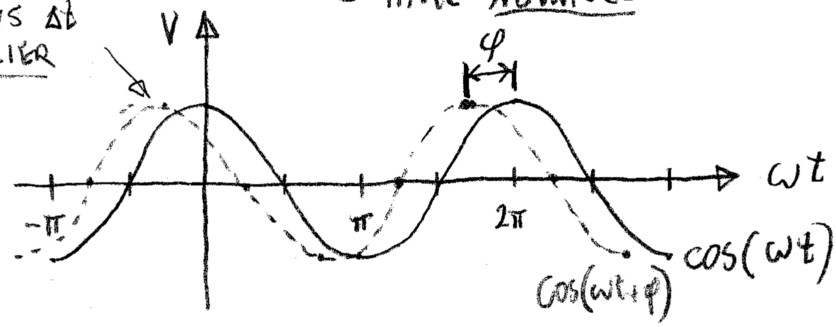
✓ COMPARE TO THE GRAPHICAL RESULT ABOVE

PHASE SHIFTS

PHASE SHIFT (RADIANS

$$V = V_0 \cos(\omega t + \varphi)$$
$$= V_0 \cos(\omega[t + \Delta t]) \; ; \quad \text{so} \quad \omega \Delta t = \varphi$$

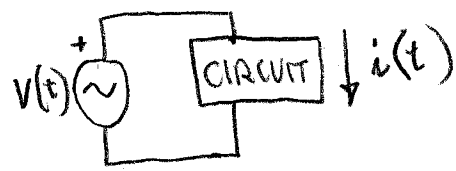
TIME ADVANCE

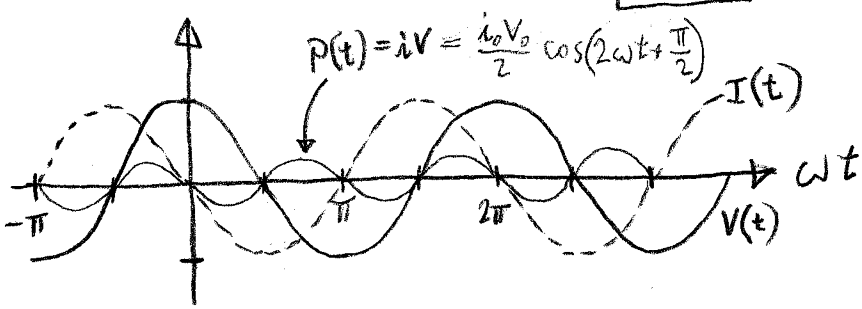


WHAT HAPPENS IF VOLTAGE & CURRENT ARE "OUT OF PHASE"?

I.E. $V = V_0 \cos(\omega t)$
$i = i_0 \cos(\omega t + \varphi)$ } DIFFERENT PHASE!

FOR EXAMPLE, $\varphi = \frac{\pi}{2}$:





P>0: CIRCUIT IS DISSIPATING POWER

P<0: CIRCUIT IS A POWER SOURCE (SENDING POWER BACK INTO THE VOLTAGE SOURCE)

Q: FOR $\varphi = \frac{\pi}{2}$, WHAT IS $\overline{P}$?

A: $\overline{P} = 0$. THERE IS NO AVERAGE POWER, BECAUSE THE POSITIVE PEAKS AND NEGATIVE VALLEYS EXACTLY CANCEL.

MORE GENERALLY,

IF $i = i_0 \cos(\omega t + \varphi)$

$v = V_0 \cos(\omega t)$

THEN

$$p = iv = i_0 V_0 (\cos \omega t \cos \varphi - \sin \omega t \sin \varphi) \cos \omega t$$

$$= i_0 V_0 \Big(\underbrace{\cos^2 \omega t}_{\frac{1}{2}(1 + \cos 2\omega t)} \cos \varphi - \underbrace{\sin \omega t \cos \omega t}_{\frac{1}{2} \sin 2\omega t} \sin \varphi\Big)$$

$$= \frac{i_0 V_0}{2} \Big[(1 + \cos 2\omega t) \cos \varphi - \sin 2\omega t \sin \varphi \Big]$$

$$= \frac{i_0 V_0}{2} \Big[\cos \varphi + \underbrace{\cos 2\omega t \cos \varphi - \sin 2\omega t \sin \varphi}_{\cos(2\omega t + \varphi)} \Big]$$

$$= \frac{i_0 V_0}{2} \Big[\cos \varphi + \cos(2\omega t + \varphi) \Big]$$

AMPLITUDE — MEAN VALUE — TIME VARIATION WITH MEAN OF ZERO

SIMILARLY, IF

$i = i_0 \cos(\omega t + \varphi_1)$, AND

$v = V_0 \cos(\omega t + \varphi_2)$, THEN

$$p = iV = \frac{i_0 V_0}{2} \Big[\cos(\varphi_2 - \varphi_1) + \cos(2\omega t + \varphi_1 + \varphi_2) \Big]$$

<u>SO</u> $\boxed{\bar{P} = \frac{i_0 V_0}{2} \cos(\varphi_2 - \varphi_1)}$

USING COMPLEX NUMBERS TO REPRESENT AC:

IF $V = V_0 \cos(\omega t + \varphi)$, WE WRITE

$$\underset{\sim}{V} = V_0\, e^{j(\omega t + \varphi)} = \underset{\sim}{V_0}\, e^{j\omega t}, \qquad \underbrace{\underset{\sim}{V_0} = V_0\, e^{j\varphi}}$$

$|\underset{\sim}{V}|$ = AMPLITUDE

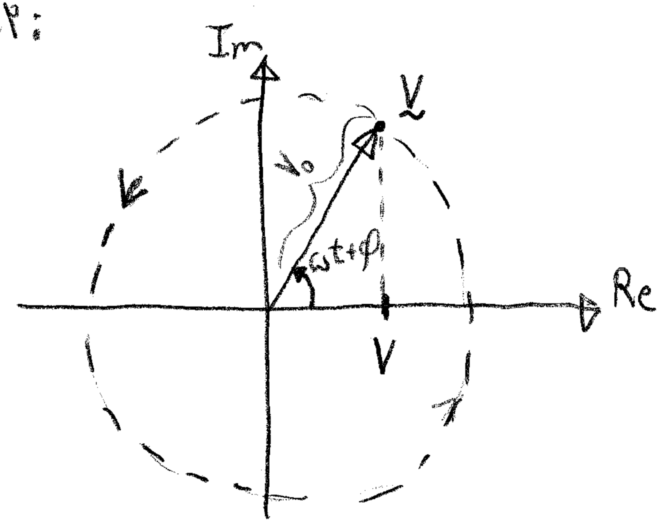
ANGULAR FREQUENCY

PHASE SHIFT

COMPLEX AMPLITUDE

THE COMPLEX AMPLITUDE ENCODES THE PHASE SHIFT, AS WELL AS THE AMPLITUDE.

GRAPHICALLY:



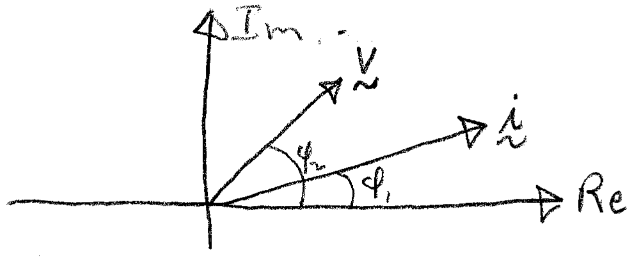
THE "PHASOR" (ARROW) HOLDS CONSTANT RADIUS V_0 AS IT ROTATES COUNTER CLOCKWISE.

RECALL THAT IN OUR STUDY OF POWER, WE FOUND

IF $i = i_0 \cos(\omega t + \varphi_1)$, $V = V_0 \cos(\omega t + \varphi_2)$, THEN

$$\overline{P} = \tfrac{1}{2} i_0 V_0 \cos(\varphi_2 - \varphi_1) = \tfrac{1}{2} i_0 V_0 \cos(\varphi_1 - \varphi_2)$$

THIS IS LIKE A DOT PRODUCT, $\vec{i} \cdot \vec{V}$.



CONSIDER: $\vec{z} \cdot \vec{z} = |z|^2 \equiv \underset{\sim}{z}^* \underset{\sim}{z}$

BY ANALOGY, TRY $\overline{P} = \text{"}\tfrac{1}{2} \vec{i} \cdot \vec{V}\text{"} = \tfrac{1}{2} \underset{\sim}{i}^* \underset{\sim}{V} = \tfrac{1}{2} i_0 e^{-j(\omega t + \varphi_1)}\, V_0\, e^{j(\omega t + \varphi_2)}$

$$= \tfrac{1}{2} i_0 V_0\, e^{j(\varphi_2 - \varphi_1)}$$

Q: IS THIS RIGHT?

A: ALMOST.

$$\boxed{\overline{P} = \frac{1}{2} Re(i^* v) = \frac{1}{2} Re(i v^*)}$$

$$= \frac{1}{2} i_0 V_0 \cos(\varphi_2 - \varphi_1) = \frac{1}{2} i_0 V_0 \cos(\varphi_1 - \varphi_2)$$

TO SEE HOW THIS "DOT" PRODUCT WORKS CONSIDER THE ORDINARY DOT PRODUCT:

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$

IN THE COMPLEX PLANE,

$$(a + jb) \cdot (c + jd) \equiv Re\left[(a+jb)^* (c+jd)\right]$$

$$= Re\left[(a-jb)(c+jd)\right] =$$

$$= Re\left[ac + bd + j\underbrace{(ad - bc)}\right]$$

$$= ac + bd \; \checkmark$$

INTERESTINGLY, THE IMAGINARY PIECE IS THE CROSS PRODUCT!