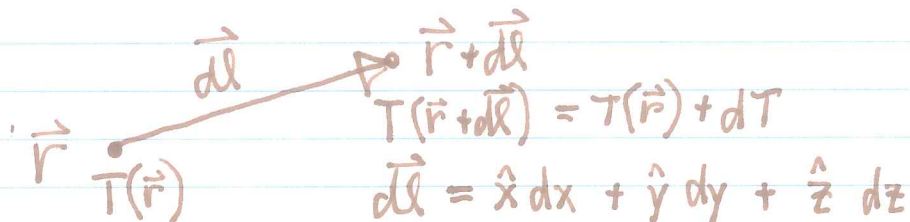


①  
 GRIFFITHS § 1.2 — DIFFERENTIATION

CONSIDER A SCALAR FUNCTION  $T(\vec{r})$ .



$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

IMAGINE THIS AS A DOT PRODUCT:

$$\begin{aligned}
 dT &= \nabla T \cdot d\vec{l} \\
 &= \underbrace{\left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)}_{\nabla \text{ ("DEL")}} T \cdot d\vec{l}
 \end{aligned}$$

$$\nabla T = \hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \text{ IS THE GRADIENT OF } T.$$

THIS VECTOR POINTS IN THE DIRECTION OF STEEPEST ASCENT;

$$dT = |\nabla T| |d\vec{l}| \cos \theta, \text{ so}$$

$$\frac{dT}{d\ell} = |\nabla T| \cos \theta.$$

MAXIMUM VALUE  $|\nabla T|$  OCCURS AT  $\theta = 0$ ,  
 WITH  $d\vec{l}$  ALIGNED TO THE GRADIENT. AT  $\theta = \pi$ ,

$$\frac{dT}{d\ell} \rightarrow -|\nabla T|.$$

(2)

EXERCISE:  $r \equiv \sqrt{x^2 + y^2 + z^2}$ . FIND  $\nabla r$ .

$$\frac{\partial r}{\partial x} = \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\therefore \nabla r = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \frac{\vec{r}}{r} = \hat{r}.$$

EXERCISE: FIND THE MINIMUM OF

$$f(x, y) = 3x^2 + 3y^2 - 2xy - 2x - 10y + 2.$$

STRATEGY: MIN ( $f$ ) OCCURS AT  $\nabla f = 0$ .

$$\frac{\partial f}{\partial x} = 6x - 2y - 2 \Rightarrow 3x - y - 1 = 0$$

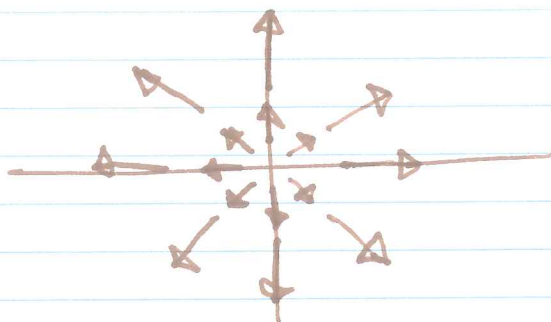
$$\frac{\partial f}{\partial y} = 6y - 2x - 10 \Rightarrow 3y - x - 5 = 0$$

$$\text{SOLUTION: } x = 1, \quad y = 2, \quad f(1, 2) = -9.$$



EXAMPLECALCULATE  $\nabla \cdot \vec{v}$ ,  $\vec{v} = \vec{r}$ .

$$\nabla \cdot \vec{v} = \nabla \cdot (x\hat{x} + y\hat{y} + z\hat{z}) = 1 + 1 + 1 = 3$$



SUPPOSE THIS REPRESENTS THE VELOCITIES OF GALAXIES IN THE UNIVERSE. WHAT DOES IT LOOK LIKE IF WE MOVE THE ORIGIN AND THE FRAME OF REFERENCE TO A DIFFERENT GALAXY?

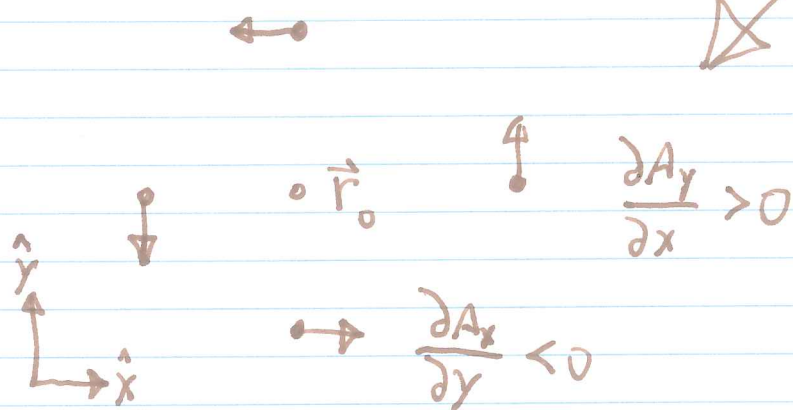
CURL

$$\nabla \times \vec{A} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right).$$

AGAIN TAKE  $\vec{A}_1 = \vec{A}(\vec{r}) - \vec{A}(\vec{r}_0)$ . SUBTRACTING  
A CONSTANT VECTOR DOES NOT CHANGE THE CURL.

LOOK IN THE NEIGHBORHOOD OF  $\vec{r}_0$  AT JUST THE  
Z-COMPONENT OF THE CURL:



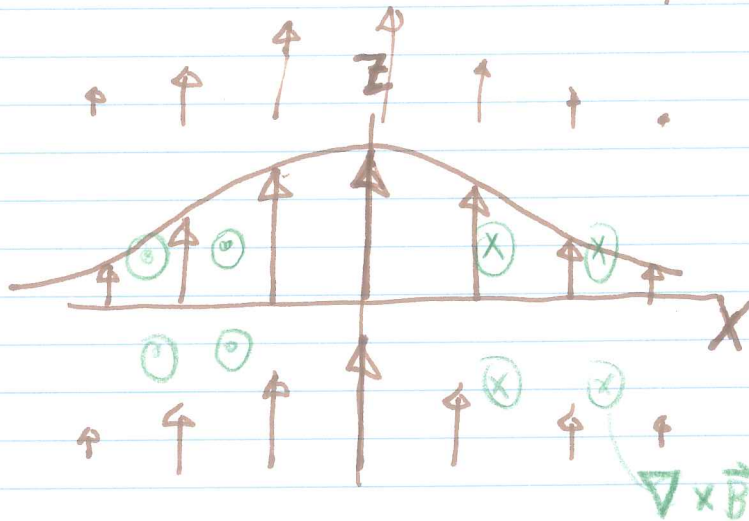
IMAGINE  $\vec{A}$  TO BE THE CURRENT IN SOME  
BODY OF WATER.  $\nabla \times \vec{A}$  ESTABLISHES THE  
ROTATION OF A SMALL OBJECT EMBEDDED  
IN THE CURRENT.

EXAMPLE FIND  $\nabla \times \vec{r}$  :

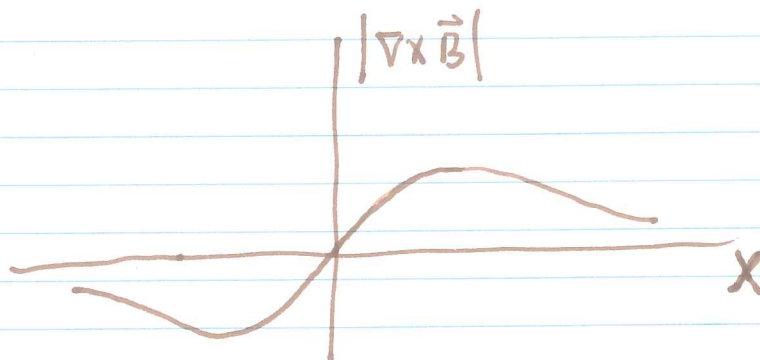
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

EXAMPLE LET  $\vec{B} = \hat{z} e^{-x^2/2}$ . FIND  $\nabla \times \vec{B}$ .

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & e^{-x^2/2} \end{vmatrix} = \hat{y} \times e^{-x^2/2}$$



VECTOR  
FIELD  
(x-z PLANE)



## PRODUCT RULES

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$$\nabla(fg) = f\nabla g + g\nabla f. \quad (\text{OBVIOUS?})$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A}.$$

$$\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f.$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}).$$

$$\star \nabla \times (f\vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f).$$

$$\begin{aligned} \nabla \times (\vec{A} \times \vec{B}) &= (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \\ &\quad + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A}). \end{aligned}$$

$$- \text{CHECK THAT } \nabla \times (\vec{A} \times \vec{B}) = -\nabla \times (\vec{B} \times \vec{A}).$$

WHAT OTHER SIMPLE CHECKS CAN WE PERFORM?

EXERCISE: PROVE  $\star$ .

## SECOND DERIVATIVES

$$\nabla \cdot (\nabla T) \equiv \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \quad \text{"LAPLACIAN"}$$

VERY COMMON OPERATOR IN PHYSICS EQUATIONS!

THERE ARE SEVERAL USEFUL SECOND DERIVATIVE IDENTITIES IN THE TEXT. THE MOST IMPORTANT ARE:

$$\left. \begin{aligned} \nabla \times (\nabla T) &= 0 \\ \nabla \cdot (\nabla \times \vec{A}) &= 0 \end{aligned} \right\} \begin{array}{l} \text{HW} \\ \text{PROBLEMS} \end{array}$$