

GRIFFITHS § 1.3 - INTEGRAL CALCULUS

①

LINE INTEGRALS

GIVEN SOME $\vec{v}(\vec{r})$ AND A PATH C ,
CALCULATE

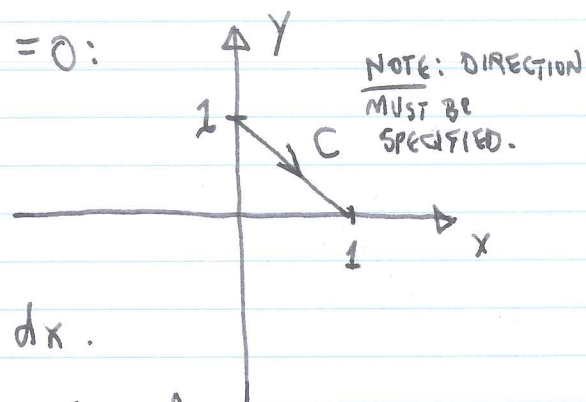
$$\int_C \vec{v} \cdot d\vec{r}.$$

IF C IS CLOSED, THEN WE WRITE $\oint_C \vec{v} \cdot d\vec{r}.$

EXAMPLE

$\vec{v} = \vec{r}$, C IN PLANE $z = 0$:

C IS DESCRIBED BY $y = 1 - x$,
 $dy = -dx$, $0 \leq x \leq 1$.



$$d\vec{r} = \hat{x} dx + \hat{y} dy = (\hat{x} - \hat{y}) dx.$$

$$\vec{v} = x\hat{x} + y\hat{y} + \cancel{z}\hat{z} = x\hat{x} + (1-x)\hat{y}.$$

$$\vec{v} \cdot d\vec{r} = \cancel{0} (x - (1-x)) dx = (2x-1) dx$$

$$\int_C \vec{v} \cdot d\vec{r} = \int_0^1 (2x-1) dx = (x^2 - x) \Big|_0^1 = 0.$$

Q: DOES THE ANSWER MAKE SENSE? DISCUSS/SKETCH.

THE STRATEGY, THEN, IS TO WRITE THE LINE INTEGRAL IN TERMS OF A SINGLE VARIABLE. FOR NOW IT IS x , y , OR z ; BUT IT MIGHT BE NECESSARY TO CHANGE COORDINATES OR INVENT SOME PARAMETER TO PARAMETRIZE C .

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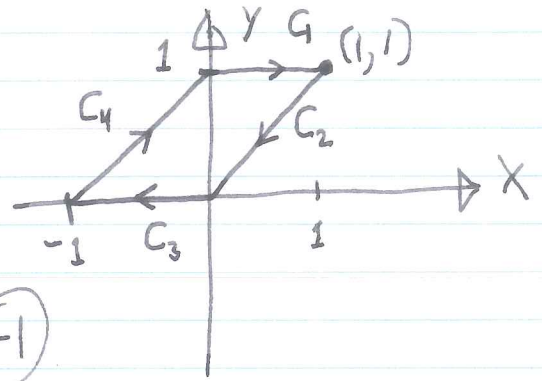
EXERCISELET $\vec{v} = x\hat{y} - y\hat{x}$. FIND $\oint \vec{v} \cdot d\vec{l}$ AS ILLUSTRATED.

$$(C_1) \quad y=1, \quad dy=0, \quad \vec{v} = -\hat{x} + x\hat{y}$$

$$d\vec{l} = \hat{x} dx + \hat{y} dy = \hat{x} dx$$

$$\vec{v} \cdot d\vec{l} = -dx$$

$$\int_{C_1} \vec{v} \cdot d\vec{l} = -\int_0^1 dx = -x \Big|_0^1 = \textcircled{-1}$$



$$(C_2) \quad y=x, \quad dx=dy, \quad \vec{v} = x\hat{y} - x\hat{x}, \quad d\vec{l} = (\hat{x} + \hat{y}) dx$$

$$\Rightarrow \int_{C_2} \vec{v} \cdot d\vec{l} = \int (x-x) dx = \textcircled{0}$$

$$(C_3) \quad y=0, \quad dy=0, \quad \vec{v} = x\hat{y} - 0\hat{x}, \quad d\vec{l} = \hat{x} dx + \hat{y} dy$$

$$\Rightarrow \vec{v} \cdot d\vec{l} = 0, \quad \int_{C_3} \vec{v} \cdot d\vec{l} \neq 0$$

$$(C_4) \quad y=1+x, \quad dx=dy, \quad \vec{v} = x\hat{y} - (1+x)\hat{x}$$

$$d\vec{l} = \hat{x} dx + \hat{y} dy = (\hat{x} + \hat{y}) dx$$

$$\vec{v} \cdot d\vec{l} = (x - (1+x)) dx = -dx$$

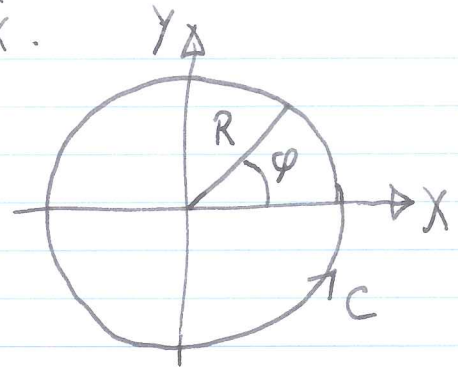
$$\int_{C_4} \vec{v} \cdot d\vec{l} = -\int_{-1}^0 dx = -[x]_{-1}^0 = -(0 - (-1))$$

$$= \textcircled{-1}$$

EXAMPLE

SAME VECTOR FIELD, $\vec{v} = x\hat{y} - y\hat{x}$.

DIFFERENT PATH, CIRCLE OF RADIUS R CENTERED AT THE ORIGIN.



$$\vec{v} = -\hat{x} \underbrace{R \sin \phi}_y + \hat{y} \underbrace{R \cos \phi}_x$$

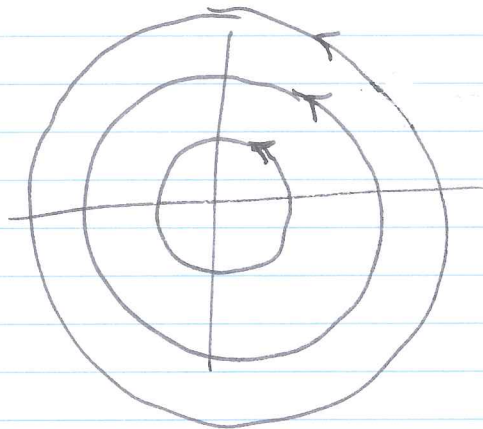
$$d\vec{l} = R d\phi \hat{\phi}; \quad \hat{\phi} = [-\hat{x} \sin \phi + \hat{y} \cos \phi]$$

$$\vec{v} \cdot d\vec{l} = R^2 d\phi [\sin^2 \phi + \cos^2 \phi] = R^2 d\phi$$

$$\oint_c \vec{v} \cdot d\vec{l} = \int_{\phi=0}^{2\pi} R^2 d\phi = \boxed{2\pi R^2}$$

NOTICE THAT $\vec{v} = r \hat{\phi}$. (SKETCH BELOW)

LOOK BACK AT THE PREVIOUS EXAMPLE — DOES THIS HELP EXPLAIN WHAT WAS HAPPENING?



INTERPRETATION

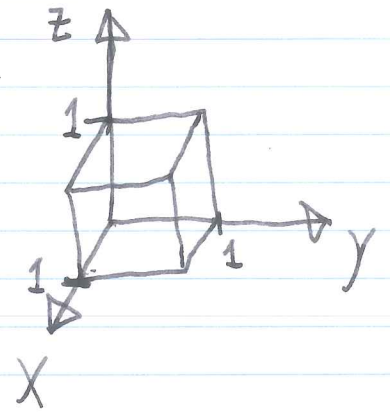
$\vec{v} = r \hat{\phi} = r \vec{\omega}$
FIELD OF CONSTANT ANGULAR VELOCITY.

SURFACE INTEGRALS

$$\int \vec{B} \cdot d\vec{a} \equiv \int (\vec{B} \cdot \hat{n}) da$$

EXAMPLE: LET $\vec{v} = \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

CALCULATE $\oint \vec{v} \cdot d\vec{a}$ OVER THE CLOSED SURFACE OF THE UNIT CUBE.



① z=1: $d\vec{a} = \hat{z} dx dy$, $\vec{r} = x\hat{x} + y\hat{y} + 1\hat{z}$

$$\vec{r} \cdot d\vec{a} = dx dy$$

$$\int_0^1 \int_0^1 \vec{r} \cdot d\vec{a} = \int_0^1 \int_0^1 dx dy = \textcircled{1}$$

② z=0: $\vec{r} \cdot d\vec{a} = 0$ (BY INSPECTION)

$$\text{SO } \int_0^0 \vec{r} \cdot d\vec{a} = \textcircled{0}$$

③ x=1: SAME AS z=1, $\int_0^1 \int_0^1 \vec{r} \cdot d\vec{a} = 1$

④ x=0: SAME AS z=0, $\int_0^0 \vec{r} \cdot d\vec{a} = 0$

⑤ y=1: SAME AS z=1, $\int_0^1 \int_0^1 \vec{r} \cdot d\vec{a} = 1$

⑥ y=0: SAME AS y=0, $\int_0^0 \vec{r} \cdot d\vec{a} = 0$

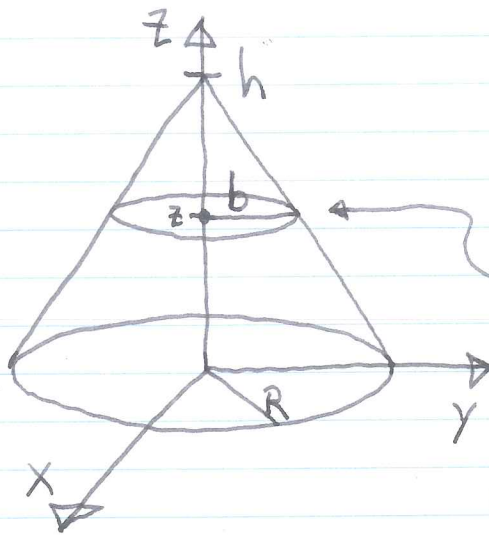
$$\therefore \oint \vec{r} \cdot d\vec{a} = \int_0^1 + \int_0^0 + \int_0^1 + \int_0^0 + \int_0^1 + \int_0^0 = \textcircled{3}$$

VOLUME INTEGRALS

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EXAMPLES

- (a) FIND THE VOLUME OF A RIGHT CIRCULAR CONE OF HEIGHT h AND BASE RADIUS R .



$$V = \int dV.$$

A CIRCLE AT HEIGHT z HAS RADIUS $b = R(1 - z/h)$.

AND AREA $A = \pi b^2 = \pi R^2(1 - 2\frac{z}{h} + \frac{z^2}{h^2})$.

LET $dV = A dz$.

$$\begin{aligned} \text{NOW } V &= \int_0^h \pi R^2 \left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right) dz \\ &= \pi R^2 \left[z - \frac{z^2}{h} + \frac{1}{3} \frac{z^3}{h^2} \right]_0^h \\ &= \pi R^2 \left[h - h + \frac{1}{3} h \right] \end{aligned}$$

$$V = \frac{1}{3} \pi R^2 h.$$

- (b) FIND $\int T dV$, $\rho \equiv \sqrt{x^2 + y^2}$, $T = R/\rho$

ON A CIRCLE OF HEIGHT z ,

$$\int T dA = \int_0^b \frac{R}{\rho} \underbrace{2\pi\rho d\rho}_{dA} = 2\pi R [\rho]_0^{R(1-z/h)} = 2\pi R^2 (1 - z/h).$$

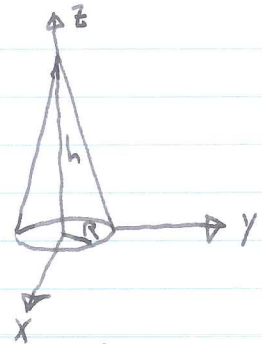
$$\begin{aligned} \int T dV &= \int_0^h [\int T dA] dz = 2\pi R^2 \int_0^h (1 - z/h) dz \\ &= 2\pi R^2 \left[z - \frac{z^2}{2h} \right]_0^h = 2\pi R^2 [h - h/2] \end{aligned}$$

$$\int T dV = \pi R^2 h.$$

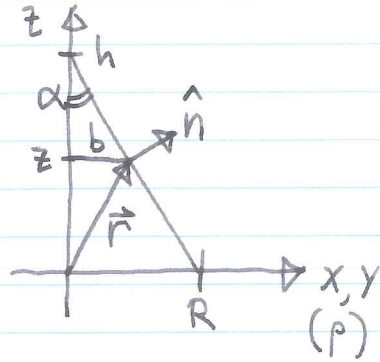
MORE SURFACE INTEGRAL EXAMPLES...

FOR THE SAME CONE, FIND $\oint \vec{r} \cdot d\vec{a}$.

- INTEGRAL OVER THE BASE IS ZERO SINCE THE \vec{r} VECTOR IS PARALLEL TO THE SURFACE.



- CONE WALLS:



UNIT NORMAL $\hat{n} = \frac{h\hat{\rho} + R\hat{z}}{\sqrt{h^2 + R^2}}$

$\vec{r} = b\hat{\rho} + z\hat{z} = R(1 - z/h)\hat{\rho} + z\hat{z}$

$\vec{r} \cdot \hat{n} = \frac{R(h-z) + zR}{\sqrt{h^2 + R^2}} = \frac{hR}{\sqrt{h^2 + R^2}}$
 CONSTANT.

$\Rightarrow \oint \vec{r} \cdot d\vec{a} = \overset{\text{BASE}}{\downarrow} 0 + \int (\vec{r} \cdot \hat{n}) da = \frac{hR}{\sqrt{h^2 + R^2}} \int da$

SURFACE AREA OF CONE

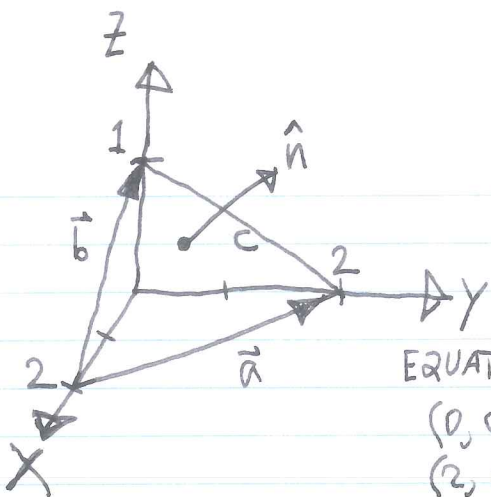
LET $da = 2\pi b \frac{\sqrt{h^2 + R^2}}{h} dz$ } DISCUSS!
 \uparrow
 $\frac{1}{\cos \alpha}$

$da = \sqrt{h^2 + R^2} \frac{2\pi R}{h} (1 - \frac{z}{h}) dz$

$\int da = \int_0^h \sqrt{h^2 + R^2} \frac{2\pi R}{h} (1 - \frac{z}{h}) dz$

$= \frac{2\pi R}{h} \sqrt{h^2 + R^2} \left[z - \frac{z^2}{2h} \right]_0^h = \pi R \sqrt{h^2 + R^2}$

$\therefore \oint \vec{r} \cdot d\vec{a} = \pi R^2 h.$



FIND $\oint \vec{r} \cdot d\vec{a}$ OVER THE TETRAHEDRON.

- THE SIDES IN THE XY, YZ, XZ PLANES ARE ALL $\hat{n} \cdot \vec{F} = 0$, SO THEY DO NOT CONTRIBUTE!

EQUATION FOR SURFACE: $z = Ax + By + C$.

$(0, 0, 1): 1 = C$
 $(2, 0, 0): 0 = 2A + C \Rightarrow A = -1/2$
 $(0, 2, 0): 0 = 2B + C \Rightarrow B = -1/2$

CONSTRUCT $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$\vec{a} = (0, 2, 0) - (2, 0, 0) = (-2, 2, 0)$
 $\vec{b} = (0, 0, 1) - (2, 0, 0) = (-2, 0, 1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -2 & 2 & 0 \\ -2 & 0 & 1 \end{vmatrix} = 2\hat{x} + (-0+2)\hat{y} + (0+4)\hat{z}$$

$$\therefore \hat{n} = \frac{2\hat{x} + 2\hat{y} + 4\hat{z}}{\sqrt{4+4+16}} = \frac{\hat{x} + \hat{y} + 2\hat{z}}{\sqrt{6}}$$

$$\vec{r} \cdot \hat{n} = (x\hat{x} + y\hat{y} + z\hat{z}) \cdot (\hat{x} + \hat{y} + 2\hat{z}) / \sqrt{6} = \frac{x+y+2z}{\sqrt{6}}$$

USING OUR EQUATION FOR THE SURFACE, $z = -\frac{1}{2}x - \frac{1}{2}y + 1$,

$$\vec{r} \cdot \hat{n} = \frac{2}{\sqrt{6}}. \quad (\text{NOTE: THIS IS THE SHORTEST DISTANCE FROM THE PLANE TO THE ORIGIN})$$

$$\therefore \oint \vec{r} \cdot d\vec{a} = \underbrace{0+0+0}_{\text{OTHER SIDES}} + \int_{abc} (\vec{F} \cdot \hat{n}) da = \frac{2}{\sqrt{6}} \int_{abc} da$$

AREA OF TRIANGLE abc

HERON'S FORMULA: $\int_{abc} da = \sqrt{s(s-a)(s-b)(s-c)}$, $s \equiv \frac{\text{PERIMETER}}{2} = \frac{a+b+c}{2}$

$a = 2\sqrt{2}$; $b = c = \sqrt{5}$; $s = \sqrt{2} + \sqrt{5}$.

$$\int_{abc} da = \sqrt{(\sqrt{2} + \sqrt{5})(\sqrt{5} - \sqrt{2})\sqrt{2}\sqrt{2}} = \sqrt{2(s-2)} = \sqrt{6}$$

$$\therefore \boxed{\oint \vec{r} \cdot d\vec{a} = 2}$$