

RECALL FROM LAST TIME

## FUNDAMENTAL THEOREM OF CALCULUS

$$\int_a^b \left( \frac{df}{dx} \right) dx = f(b) - f(a)$$

## GRADIENT THEOREM

$$\int_{\vec{A}}^{\vec{B}} (\nabla T) \cdot d\vec{l} = T(\vec{B}) - T(\vec{A})$$

COROLLARIES:

- THE ABOVE INTEGRAL IS PATH INDEPENDENT.
- $\oint (\nabla T) \cdot d\vec{l} = 0$ .

## DIVERGENCE THEOREM (GAUSS'S THM, GREEN'S THM)

$$\int_V (\nabla \cdot \vec{v}) dV = \oint_S \vec{v} \cdot d\vec{a}$$

EXAMPLE: WHAT IS  $\oint_S (\nabla \times \vec{A}) \cdot d\vec{A}$  ?TODAY: STOKES' THM (CURL THEOREM) GRIFFITHS § 1.3.5

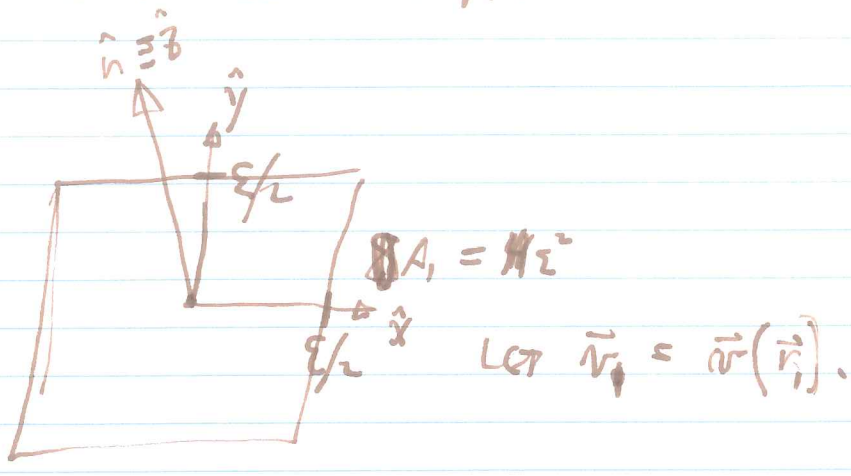
$$\boxed{\int_A (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_C \vec{v} \cdot d\vec{l}}$$

IMPORTANT: DIRECTION OF  $d\vec{a} = \hat{n} da$  GIVEN BY  
RIGHT HAND RULE ON CONTOUR C.

IN THE NEIGHBORHOOD OF SOME POINT  $\vec{r}_1$ ,

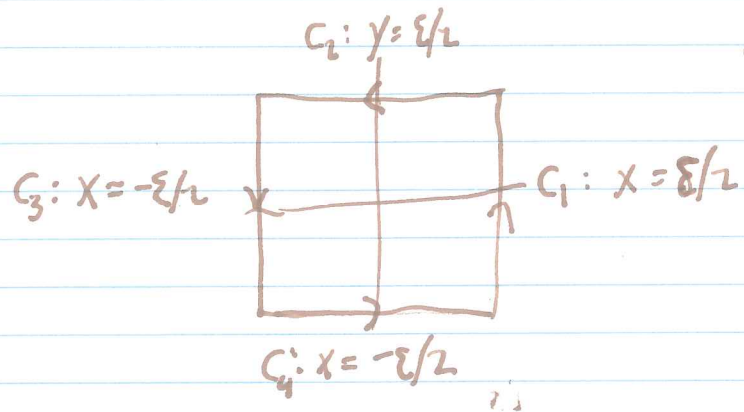
CONSIDER A SMALL <sup>SQUARE</sup> BIT OF AREA  $A_1$  ON A LARGER SURFACE  $S$ . WITH NORMAL VECTOR  $\hat{n}$ .

CONSTRUCT A COORDINATE SYSTEM WITHIN  $A_1$ , ORIGIN AT CENTER.



APPROX LET  $\epsilon$  BE SMALL ENOUGH THAT DERIVATIVES OF  $\vec{v}(\vec{r})$  ARE CONSTANT. THUS  $\vec{v} = \vec{v}_1 + x \frac{\partial \vec{v}}{\partial x} + y \frac{\partial \vec{v}}{\partial y}$ . WITHIN  $A_1$ .

WE SEEK TO CALCULATE  $\oint_{\partial A_1} \vec{v} \cdot d\vec{l}$ , ON THE BOUNDARY OF  $A_1$ .



$$\oint \vec{v} \cdot d\vec{l} = \oint \vec{v}_1 \cdot d\vec{l} + \oint \left( x \frac{\partial \vec{v}}{\partial x} + y \frac{\partial \vec{v}}{\partial y} \right) \cdot d\vec{l}$$

$C_1$ : ~~on~~  $x = \epsilon/2$ ,  $d\vec{l} = \hat{y} dy$ ,  ~~$\frac{\epsilon}{2}$~~

$$\int_{C_1} \left( x \frac{\partial \vec{v}}{\partial x} + y \frac{\partial \vec{v}}{\partial y} \right) \cdot d\vec{l} = \int_{-\epsilon/2}^{\epsilon/2} \left( \frac{\epsilon}{2} \frac{\partial v_x}{\partial x} + y \frac{\partial v_x}{\partial y} \right) dy =$$

$$= \left[ \frac{\epsilon}{2} \frac{\partial v_x}{\partial x} y + \frac{1}{2} y^2 \frac{\partial v_x}{\partial y} \right]_{-\epsilon/2}^{\epsilon/2} = \frac{\epsilon^2}{2} \frac{\partial v_x}{\partial x}$$

$C_2: y = \epsilon/2, \vec{dl} = \hat{x} dx$

$$\int_{C_2} \left( x \frac{\partial v_x}{\partial x} + y \frac{\partial v_x}{\partial y} \right) \cdot \vec{dl} = \int_{\epsilon/2}^{-\epsilon/2} \left( x \frac{\partial v_x}{\partial x} + \frac{\epsilon}{2} \frac{\partial v_x}{\partial y} \right) dx$$

$$= \left[ \frac{1}{2} x^2 \frac{\partial v_x}{\partial x} + \frac{\epsilon}{2} \frac{\partial v_x}{\partial y} x \right]_{\epsilon/2}^{-\epsilon/2}$$

$$= - \frac{\epsilon^2}{2} \frac{\partial v_x}{\partial y}$$

$C_3: \text{SAME AS } C_1, \int_{C_3} = \frac{\epsilon^2}{2} \frac{\partial v_y}{\partial x}$

$C_4: \text{SAME AS } C_2, \int_{C_4} = - \frac{\epsilon^2}{2} \frac{\partial v_x}{\partial y}$

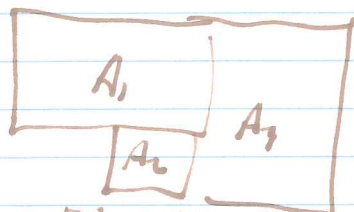
$$\therefore \oint \vec{v} \cdot \vec{dl} = \frac{\epsilon^2}{2} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$$

$$= (\nabla \times \vec{v}) \cdot \hat{z} A = (\nabla \times \vec{v}) \cdot \hat{n} \int_{A_1} da$$

SINCE  $\nabla \times \vec{v} = \text{CONST.}$ ,

$$\oint \vec{v} \cdot \vec{dl} = \int_{A_1} (\nabla \times \vec{v}) \cdot \hat{n} da = \int_{A_1} (\nabla \times \vec{v}) \cdot \vec{da}$$

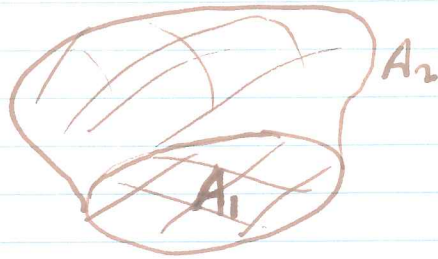
NOW, NOTICE THAT LARGER AREAS, <sup>OF ARBITRARY SHAPE</sup> CAN BE CONSTRUCTED BY JOINING SMALLER ONES:



THE GAU & DIV THEOREMS CAN BE DEMONSTRATED SIMILARLY.

# COROLLARIES OF STOKES' THEOREM

$\int_A (\nabla \times \vec{v}) \cdot d\vec{a}$  IS INDEPENDENT OF SURFACE FOR ALL SURFACES BOUNDED BY THE SAME PATH.



$\oint (\nabla \times \vec{v}) \cdot d\vec{a} = 0$  ("SHRINKING MOUTH" ARGUMENT)

CONNECTION TO GAUSS'S THEOREM:

$$\oint (\nabla \times \vec{v}) \cdot d\vec{a} = \int_V \nabla \cdot (\nabla \times \vec{v}) dV = \int_V 0 dV = 0 \checkmark$$

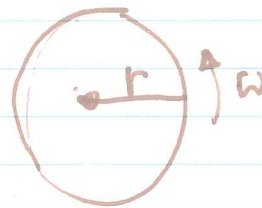
CONNECTION TO GRADIENT THEOREM:

$$\begin{aligned} \oint (\nabla T) \cdot d\vec{l} &= 0 \leftarrow \text{COROLLARY OF } \nabla \text{ THM.} \\ &\quad \swarrow \text{STOKES} \\ &= \int [\nabla \times (\nabla T)] \cdot d\vec{a} = \int \vec{0} \cdot d\vec{a} = 0 \checkmark \end{aligned}$$

EXAMPLE/  
EXERCISE

5

LET  $\vec{v}(\vec{r})$  REPRESENT THE VELOCITY OF A DISK OF RADIUS  $R$ , SPINNING AT ANGULAR FREQUENCY  $\omega$ .



(a) FIND  $\nabla \times \vec{v}$

(b) CALCULATE  $\oint \vec{v} \cdot d\vec{l}$  AROUND THE DISK'S CIRCUMFERENCE.

(c) ~~CALCULATE~~ VERIFY STOKES' THM BY CALCULATING  $\int (\nabla \times \vec{v}) \cdot d\vec{a}$ .

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$$\vec{v} = \omega r \hat{\phi} = \omega(x\hat{y} - y\hat{x})$$

(a) 
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y\omega & x\omega & 0 \end{vmatrix} = 2\omega \hat{z}$$

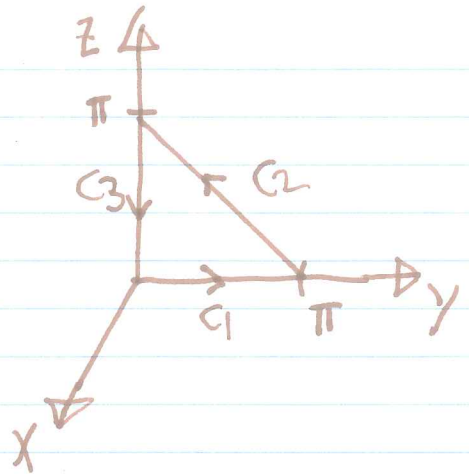
(b) 
$$\oint \vec{v} \cdot d\vec{l} = \int_0^{2\pi} (r\omega \hat{\phi}) \cdot (r\hat{\phi} d\phi) = 2\pi r^2 \omega$$

(c) 
$$\int (\nabla \times \vec{v}) \cdot d\vec{a} = 2\omega \int da = 2\pi r^2 \omega \checkmark$$

EXERCISE/  
EXAMPLE

$$\vec{v} = \hat{x} \cos z + \hat{y} \sin z + \hat{z}$$

VERIFY STOKES' THEOREM FOR THE ILLUSTRATED PATH.



$$C1: \vec{v} = \hat{x} + \hat{z}, \quad d\vec{l} = \hat{y} dy$$

$$\int_{C1} \vec{v} \cdot d\vec{l} = 0$$

$$C2: z = \pi - y, \quad d\vec{l} = \hat{z} dz + \hat{y} dy = (\hat{z} - \hat{y}) dz$$

LET'S PUT EVERYTHING IN TERMS OF z!

$$\vec{v} \cdot d\vec{l} = (1 - \sin z) dz$$

$$\int_{C2} \vec{v} \cdot d\vec{l} = \int_0^\pi (1 - \sin z) dz = [z + \cos z]_0^\pi = \pi - 1$$

$$C3: y=0, \quad d\vec{l} = dz \hat{z}, \quad \vec{v} \cdot d\vec{l} = dz$$

$$\int_\pi^0 dz = -\pi$$

$$\therefore \oint \vec{v} \cdot d\vec{l} = -2$$

NOW,

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos z & \sin z & 1 \end{vmatrix} = -\hat{x} \cos z - \hat{y} \sin z, \quad d\vec{n} = \hat{x} dy dz$$

$$\text{SO } (\nabla \times \vec{v}) \cdot d\vec{n} = -\cos z dy dz$$

$$\begin{aligned} \int (\nabla \times \vec{v}) \cdot d\vec{n} &= \int_{y=0}^\pi \left[ \int_{z=0}^{\pi-y} -\cos z dz \right] dy = \int_0^\pi [-\sin z]_0^{\pi-y} dy \\ &= \int_0^\pi \sin(y-\pi) dy = -\cos(y-\pi) \Big|_0^\pi = -[1+1] = -2 \end{aligned}$$