

## THEORY OF VECTOR FIELDS (GRIFFITHS § 1.6)

Q: IF I KNOW  $\vec{B}(\vec{r}) = \nabla \times \vec{A}$ , AND  
 $\rho(\vec{r}) = \nabla \cdot \vec{A}$ ,

DO I THEN KNOW  $\vec{A}$ ?

A: SUPPOSE  $\vec{A}'$  HAS THE SAME CURL & DIVERGENCE AS  $\vec{A}$ . SUPPOSE ALSO THAT

$$\vec{A}_0 = \vec{A}' - \vec{A}.$$

IS IT POSSIBLE THAT  $\vec{A}_0 \neq 0$ ?

$$\nabla \cdot \vec{A}_0 = \nabla \cdot (\vec{A}' - \vec{A}) = (\nabla \cdot \vec{A}') - (\nabla \cdot \vec{A}) = \rho - \rho = 0.$$

$$\nabla \times \vec{A}_0 = \nabla \times (\vec{A}' - \vec{A}) = (\nabla \times \vec{A}') - (\nabla \times \vec{A}) = \vec{B} - \vec{B} = 0.$$

THERE IS A FAMILY OF <sup>NONZERO</sup> VECTOR FIELDS THAT SATISFY BOTH EQUATIONS:

$$\text{LET } \vec{A}_0 = \nabla u, \quad \nabla^2 u = 0. \quad \checkmark$$

$$\Rightarrow \nabla \times \vec{A}_0 = \nabla \times (\nabla u) = 0, \quad \nabla \cdot \vec{A}_0 = \nabla \cdot \nabla u = \nabla^2 u = 0$$

SOLUTIONS TO LAPLACE'S EQUATION,  $\nabla^2 u = 0$ , CAN BE COMPLETELY DETERMINED FROM THE VALUE OF  $u$  EVERYWHERE ON THE SURFACE BOUNDING THE DOMAIN. THAT ALLOWS FOR A VERY LARGE NUMBER OF SOLUTIONS.

SEE PROOF IN GRIFFITHS APPENDIX B.

HELMHOLTZ THEOREM:

EVERY VECTOR FIELD  $\vec{F}$  CAN BE WRITTEN AS  
 $\vec{F} = \nabla U + \nabla \times W.$  \* \*\*

CONSEQUENTLY:

THEOREM 1 ("IRROTATIONAL" FIELDS)

- $\nabla \times \vec{F} = 0,$
- $\vec{F} = \nabla U,$
- $\int_a^b \vec{F} \cdot d\vec{l}$  IS PATH INDEPENDENT,
- $\oint \vec{F} \cdot d\vec{l} = 0.$

IF ONE OF THESE IS TRUE, THEY ARE ALL TRUE.

Q: WHICH OF THE ABOVE BULLETS CAN YOU PROVE FROM WHICH OTHERS?

THEOREM 2 ("SOLENOIDAL" FIELDS)

- $\nabla \cdot \vec{F} = 0,$
- $\vec{F} = \nabla \times \vec{W},$
- $\int \vec{F} \cdot d\vec{a}$  DEPENDS ONLY ON THE BOUNDING PATH,
- $\oint \vec{F} \cdot d\vec{a} = 0.$

IF ONE IS TRUE, THEY ARE ALL TRUE.

Q: WHICH POINTS CAN YOU PROVE FROM WHICH OTHERS?

\* GRIFFITHS WRITES  $-\nabla U$ , WHICH IS TRADITIONAL IN ELECTROSTATICS.  
\*\* ACTUALLY, THE HELMHOLTZ THEOREM CARRIES RESTRICTIONS ON  $\vec{F}$  AS  $r \rightarrow \infty$ , BUT THE RESULT IS GENERALLY TRUE.

EXERCISE

- IS THE FOLLOWING VECTOR FIELD IRROTATIONAL, SOLENOIDAL, OR BOTH?

$$\vec{v} = x\hat{x} + y\hat{y} - 2z\hat{z} \quad (\text{BOTH})$$

- FIND A SCALAR POTENTIAL,  $\vec{v} = \nabla u$

$$\left( \frac{1}{2}x^2 + \frac{1}{2}y^2 - z^2 \right)$$

$$\left( = \frac{1}{2}s^2 - z^2 \right)$$

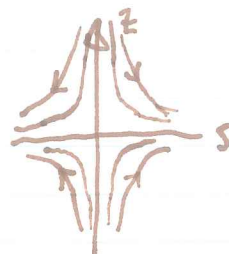
- ! • FIND A VECTOR POTENTIAL,  $\vec{v} = \nabla \times \vec{A}$

NOTE THAT  $\vec{v} = s\hat{s} - 2z\hat{z}$ .

- THIS IS AXISYMMETRIC AND HAS NO  $\phi$ -COMPONENT.

- CONSEQUENTLY, ASSUME:  $A_z = 0$ , (DISCUSS!)

$$\partial/\partial\phi = 0 \quad (\text{DISCUSS!})$$



$$\nabla \times \vec{A} = s\hat{s} - 2z\hat{z}$$

$$\Rightarrow \hat{\phi}: \frac{\partial A_s}{\partial z} = \frac{\partial A_z}{\partial s} \rightarrow \text{TRY } A_s = A_z = 0$$

$$\hat{s}: s = \frac{1}{s} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \rightarrow \text{TRY } A_\phi = -sz$$

$$\hat{z}: -2z = \frac{1}{s} \left[ \frac{\partial}{\partial s} (sA_\phi) - \frac{\partial A_s}{\partial \phi} \right] \rightarrow \text{AGAIN, } A_\phi = -sz!$$

$$\boxed{\vec{A} = -sz \hat{\phi}}$$