

GRIFFITHS §2.4 — WORK & ENERGY

GIVEN $V(\vec{r})$, THE POTENTIAL ENERGY OF A CHARGE Q AT \vec{r} IS

$$U = Q V(\vec{r}).$$

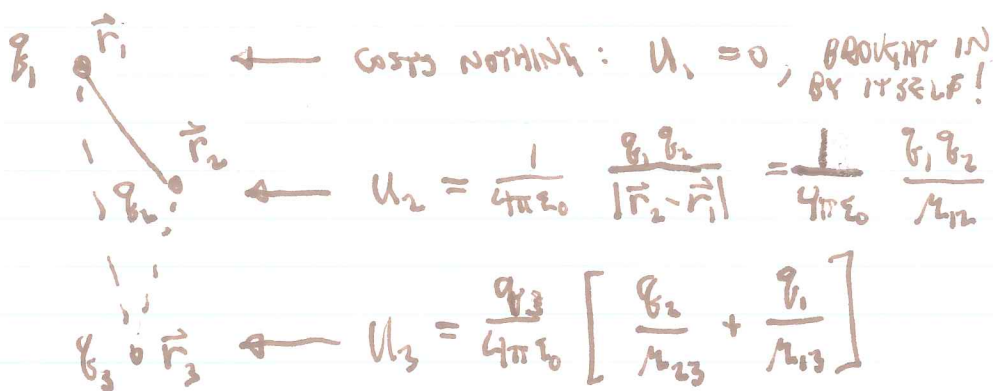
(RECALL THE "BOWL" EXAMPLE)

OPTIONAL EXAMPLE

HARMONIC MOTION OF A CHARGE IN A SOLID SPHERE WITH CONSTANT ρ .

BUT WHAT HAPPENS WHEN WE ASSEMBLE AN ARRANGEMENT OF CHARGES, q_1, q_2, q_3, \dots AT $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots$?

IMAGINE BRINGING IN THESE CHARGES ONE AT A TIME FROM INFINITY.



THE WORK DONE TO ASSEMBLE OUR CHARGE CONFIGURATION IS:

$$W = \sum_{i=1}^N U_i = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{j \neq i+1}^N \frac{q_i q_j}{r_{ij}}$$

Q: SHOULD THE ORDER OF ASSEMBLY MATTER? DISCUSS.

A: $W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^N \sum_{j \neq i} \frac{q_i q_j}{r_{ij}}$ (NO!)

TO RECAP:

WE CONSIDERED THE ENERGY OF AN ASSEMBLAGE OF POINT CHARGES, AND FOUND

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^n q_i \left(\sum_{j \neq i} \frac{q_j}{r_{ij}} \right).$$

THIS CAN BE WRITTEN

$$W = \frac{1}{2} \sum_{i=1}^n q_i V(\vec{r}_i). \quad \text{--- DISCUSS!}$$

FROM THIS FORMULA, WE WILL GENERALIZE TO A CONTINUOUS CHARGE DISTRIBUTION:

$$W = \frac{1}{2} \int V dq = \begin{cases} \frac{1}{2} \int V \rho dV \\ \frac{1}{2} \int V \sigma da \\ \frac{1}{2} \int V \lambda dl \end{cases}$$

THE DOMAIN OF THE INTEGRAL MUST INCLUDE ALL OF THE CHARGE.

USING $\rho = \epsilon_0 \nabla \cdot \vec{E}$,

$$W = \frac{\epsilon_0}{2} \int V (\nabla \cdot \vec{E}) dV$$

BUT $\nabla \cdot (\vec{E}V) = V(\nabla \cdot \vec{E}) + \vec{E} \cdot \nabla V = V \nabla \cdot \vec{E} - E^2$, SO

$$W = \frac{1}{2} \epsilon_0 \int [\nabla \cdot (\vec{E}V) + E^2] dV$$

USING THE DIVERGENCE THEOREM,

$$W = \frac{1}{2} \epsilon_0 \left[\int_V E^2 dV + \oint_S V \vec{E} \cdot d\vec{a} \right]$$

IF WE EXPAND THE DOMAIN TO $r \rightarrow \infty$, $\oint_S V \vec{E} \cdot d\vec{a} \propto \oint_S \frac{1}{r} \frac{1}{r^2} r^2 d\Omega \rightarrow 0$.
 \uparrow
 depends
 \downarrow

$$\therefore \boxed{W = \frac{1}{2} \epsilon_0 \int_V E^2 dV} \Rightarrow \frac{1}{2} \epsilon_0 E^2 \text{ IS AN ENERGY DENSITY.}$$

Q: WHAT IS THE ENERGY REQUIRED TO "ASSEMBLE" A POINT CHARGE — FOR INSTANCE, AN ELECTRON?

A: $V = \frac{-e}{4\pi\epsilon_0 r}$, $\rho = -e \delta(\vec{r})$
 (OR WE COULD USE THE FIELD, $\int \frac{1}{2} \epsilon_0 E^2 dV$)

$$W = \frac{1}{2} \int \rho V dV \propto \int \frac{1}{r} \delta(\vec{r}) dV \rightarrow \infty!$$

NOTE THAT WE IGNORED THE "SELF ENERGY" OF POINT CHARGES IN OUR DERIVATION. FOR A CONTINUOUS CHARGE DISTRIBUTION, IT ISN'T A PROBLEM.

EXAMPLE FIND THE SELF-ENERGY OF A SPHERICAL SHELL OF RADIUS R AND CHARGE q .

$$\sigma = \frac{q}{4\pi R^2}, \quad V = \frac{q}{4\pi\epsilon_0 r}, \quad r \geq R$$

$$\begin{aligned} W &= \frac{1}{2} \int \sigma V da = \frac{1}{2} \sigma V \int da \\ &= \frac{1}{2} \left(\frac{q^2}{16\pi^2 \epsilon_0 R^3} \right) 4\pi R^2 \\ &= \frac{q^2}{8\pi \epsilon_0 R} \end{aligned}$$

Q: HOW BIG WOULD AN ELECTRON BE IF ITS REST MASS WAS EQUIVALENT TO ~~FROM~~ ^{ITS} ELECTROSTATIC ENERGY?

A: $m_e c^2 = \frac{e^2}{8\pi \epsilon_0 R} \Rightarrow R = \frac{e^2}{8\pi \epsilon_0 m_e c^2} = \boxed{1.4 \times 10^{-15} \text{ m}}$

$$e = 1.6 \times 10^{-19} \text{ C}$$

$$\epsilon_0 = 8.9 \times 10^{-12} \text{ m}^{-3} \text{ kg}^{-1} \text{ s}^2 \text{ coul}^2$$

$$m_e = 9.1 \times 10^{-31} \text{ kg}$$

$$c = 3.0 \times 10^8 \text{ m/s}$$

Q: IF WE HAVE BEEN GIVEN $\rho_1(\vec{r})$ AND $\rho_2(\vec{r})$,
WE CAN CALCULATE

$$V_1 = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_1(\vec{r}')}{r} dV'$$

$$\vec{E}_1 = -\nabla V_1,$$

AND SIMILARLY \vec{E}_2 AND V_2 .

WHAT ARE V_{TOT} AND \vec{E}_{TOT} FOR $\rho_{TOT} = \rho_1 + \rho_2$?

A: $V_{TOT} = V_1 + V_2, \quad \vec{E}_{TOT} = \vec{E}_1 + \vec{E}_2.$

(PRINCIPLE OF SUPERPOSITION)

Q: NOW, SUPPOSE WE CALCULATE W_1 & W_2 .

DOES $W_{TOT} = W_1 + W_2$?

A: $W_{TOT} = \frac{1}{2} \epsilon_0 \int E_{TOT}^2 dV = \frac{1}{2} \epsilon_0 \int |\vec{E}_1 + \vec{E}_2|^2 dV$

$$= \frac{1}{2} \epsilon_0 \int (\vec{E}_1 + \vec{E}_2) \cdot (\vec{E}_1 + \vec{E}_2) dV$$

$$= \frac{1}{2} \epsilon_0 \int [E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2] dV$$

$$= W_1 + W_2 + \epsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) dV$$

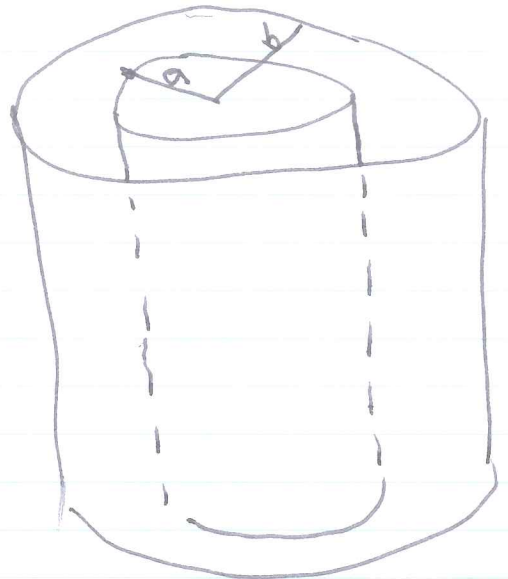
THIS COULD BE MORE OR LESS THAN $W_1 + W_2$.

→ GIVE AN EXAMPLE WHERE $W_{TOT} > W_1 + W_2$.

→ GIVE AN EXAMPLE WHERE $W_{TOT} < W_1 + W_2$.

EXAMPLE

RECALL THE HW PROBLEM
WITH $\vec{E} = \begin{cases} A \hat{s}, & a < s < b; \\ 0, & \text{ELSEWHERE.} \end{cases}$



→ WHAT IS THE ENERGY STORED
IN THIS FIELD? I.E.,
CALCULATE

$$\frac{1}{2} \epsilon_0 \int E^2 dV \quad (\text{OVER ALL SPACE}).$$

(better limit this to a fixed height)

→ ~~NO~~ FIND THE POTENTIAL $V(s)$

→ FIND THE ENERGY REQUIRED TO ASSEMBLE THE CHARGE
CONFIGURATION (AS FOUND IN THE HW SOLUTION),

$$W = \frac{1}{2} \int V dq$$