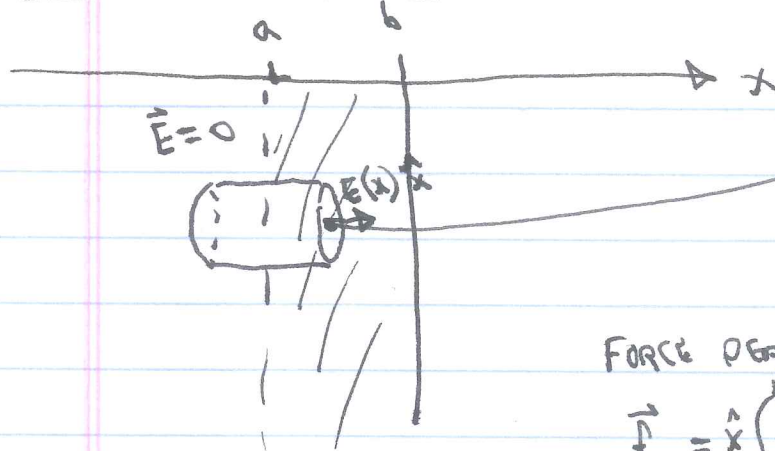


FORCE ON A CONDUCTOR § 2.5.3



$$E = \frac{\sigma}{\epsilon_0} \quad (\text{GAUSS'S LAW})$$

$$\sigma(x) = \int_a^x \rho(x') dx', \quad \sigma(b) = \int_a^b \rho(x') dx'$$

FORCE PER UNIT AREA:

$$\vec{f} = \hat{x} \int_a^b \rho(x') E(x) dx$$

$$= \frac{\hat{x}}{\epsilon_0} \int_a^b \rho(x) \left(\int_a^x \rho(x') dx' \right) dx$$

Under = ur-Grdn INTEGRATION BY PARTS: $\rho(x) dx = dv$

$$v = \int \rho(x) dx \rightarrow \int_a^x \rho(x') dx'$$

$$u = \int_a^x \rho(x') dx'$$

$$du = \rho(x) dx$$

$$\vec{f} = \frac{\hat{x}}{\epsilon_0} \left[\left(\int_a^b \rho(x') dx' \right) \left(\int_a^x \rho(x') dx' \right) \Big|_a^b - \int_a^b \rho(x) \left(\int_a^x \rho(x') dx' \right) dx \right]$$

$$\frac{\hat{x}}{\epsilon_0} \int_a^b \rho(x) \left(\int_a^x \rho(x') dx' \right) dx = \frac{\hat{x}}{\epsilon_0} \left[\left(\int_a^b \rho(x) dx \right) \left(\int_a^b \rho(x') dx' \right) - \int_a^b \rho(x) \left(\int_a^x \rho(x') dx' \right) dx \right]$$

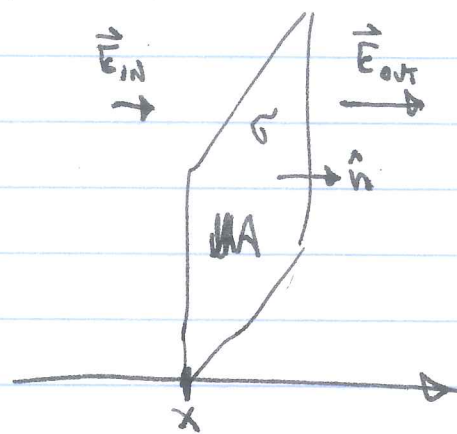
$$\therefore 2\vec{f} = \frac{\hat{x}}{\epsilon_0} \left(\int_a^b \rho(x') dx' \right)^2 = \frac{\sigma^2 \hat{x}}{\epsilon_0}$$

$$\boxed{\vec{f} = \frac{\sigma^2 \hat{n}}{2 \epsilon_0}}$$

INDEPENDENT OF HOW THE CHARGE σ IS DISTRIBUTED, ALONG THE SURFACE NORMAL.

ENERGY ARGUMENT

FORGET CONDUCTORS FOR A MOMENT. THINK OF A SURFACE CHARGE σ ; AREA ΔA



GAUSS'S LAW: $E_{out} - E_{in} = \frac{\sigma}{\epsilon_0}$

~~IF THE SURFACE WERE TO MOVE~~

HOW MUCH DOES THE ENERGY CHANGE IF I MOVE MY SURFACE dx TO THE RIGHT?

$$dW = \underbrace{\Delta A dx}_{\text{volume}} \left(\frac{1}{2} \epsilon_0 E_{in}^2 - \frac{1}{2} \epsilon_0 E_{out}^2 \right)$$

$$\frac{dW}{\Delta A dx} = \frac{1}{2} \epsilon_0 (E_{in}^2 - E_{out}^2)$$

Q: WHAT IS $\frac{dW}{\Delta A dx}$?

A: A FORCE PER AREA

Q: WHAT DIRECTION IS POSITIVE $\frac{dW}{\Delta A dx}$?

A: ~~TO THE RIGHT~~ $-\hat{n}$ (TO THE LEFT)

$$\vec{f} = -\frac{1}{2} \epsilon_0 (E_{in}^2 - E_{out}^2) \hat{n} = \frac{1}{2} \epsilon_0 (E_{out}^2 - E_{in}^2) \hat{n}$$

THIS FORCE IS AN ELECTROSTATIC TENSION. THE CHARGE IS "PULLED" INTO THE STRONGER FIELD REGION.

CONDUCTOR: $E_{in} = 0$, so $\vec{f} = \frac{1}{2} \epsilon_0 E_{out}^2 \hat{n}$

Q: ~~WHAT IS~~ CAN A CONDUCTOR FEEL A FORCE IN $-\hat{n}$ DIRECTION?
A: NO!

APPLYING OUR RESULT TO A CONDUCTOR,

DO OUR RESULTS MATCH?

$$E_{out} = \frac{\sigma}{\epsilon_0}, \text{ so YES,}$$

$$\vec{f} = \frac{1}{2} \epsilon_0 E_{out}^2 \hat{n} = \frac{\sigma^2 \hat{n}}{2\epsilon_0}$$

FORCE ON A CONDUCTOR

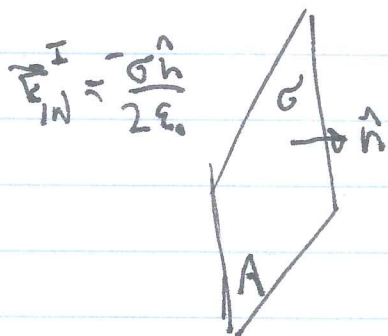
THIRD DERIVATION

NOTICE THAT WE COULD ALSO WRITE

$$\vec{f} = \frac{1}{2} \sigma E \hat{n}.$$

IT IS AS IF ONLY HALF THE CHARGE EXPERIENCES E , OR ONLY HALF OF E INFLUENCES THE CHARGE. THIS IS ACTUALLY EASY TO UNDERSTAND.

IMAGINE AN ISOLATED SURFACE CHARGE σ :



$$\vec{E}_{in}^I = -\frac{\sigma \hat{n}}{2\epsilon_0}$$

$$\vec{E}_{out}^I = \frac{\sigma \hat{n}}{2\epsilon_0}$$

SINCE σ IS ISOLATED, AND IT CAN'T EXERT A FORCE ON ITSELF, $\vec{f}^I = 0$.

NOW IMPOSE AN EXTERNAL FIELD, $\vec{E}_{EXT} = E_{EXT} \hat{n}$.

THE FORCE IS CLEARLY $\vec{f} = \sigma \vec{E}_{EXT}$

NOW IN THE CASE OF THE CONDUCTOR,

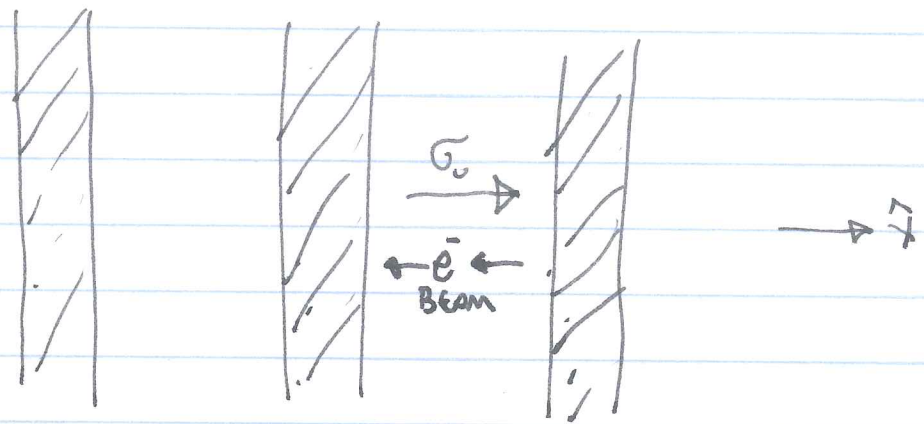
$$\vec{E}_{in} = \vec{E}_{in}^I + \vec{E}_{EXT} = 0 \Rightarrow \vec{E}_{EXT} = \frac{\sigma \hat{n}}{2\epsilon_0}$$

$$\therefore \vec{f} = \frac{\sigma^2 \hat{n}}{2\epsilon_0} \quad \checkmark$$

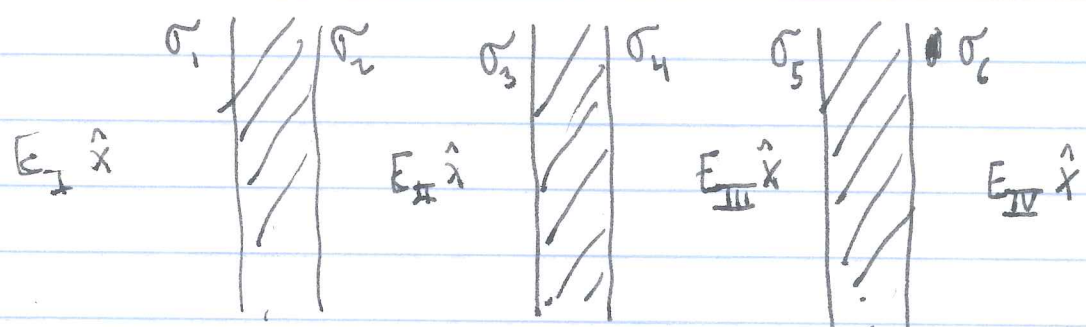
THIS DERIVATION IS MORE SIMILAR TO THAT IN GRIFFITHS.

EXAMPLE

3 CONDUCTING PLATES, ALL NEUTRAL, $V = \sigma = 0$ EVERYWHERE.



NOW A CHARGE σ_0 IS TRANSFERRED FROM THE MIDDLE PLATE TO THE RIGHT PLATE. \downarrow HOW DOES THE CHARGE DISTRIBUTE ITSELF? WHAT IS THE FORCE ON EACH PLATE? WHAT ARE THE FIELDS?



~~OR~~ $\sigma_1 + \sigma_2 = 0$ (1) $\sigma_3 + \sigma_4 = -\sigma_0$ (2) $\sigma_5 + \sigma_6 = \sigma_0$ (3)

Q: WHAT ELSE DO I KNOW?

A: $\vec{E} = 0$ IN ALL 3 CONDUCTORS! \Rightarrow MORE EQUATIONS:

(4) $\sigma_1 - (\sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6) = 0 \Rightarrow \sigma_1 - \sigma_2 = 0 \Rightarrow \boxed{\sigma_1 = \sigma_2 = 0}$

(5) ~~$\sigma_1 + \sigma_2 + \sigma_3 - (\sigma_4 + \sigma_5 + \sigma_6) = 0$~~ $\Rightarrow -\sigma_0 - 2\sigma_4 - \sigma_0 = 0 \Rightarrow \boxed{\sigma_4 = 0}$

$\boxed{\sigma_3 = 0}$

(6) $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 - \sigma_6 = 0 \Rightarrow -\sigma_0 + (\sigma_5 + \sigma_6) - 2\sigma_0 = 0$

$\Rightarrow \boxed{\sigma_6 = 0}$
 $\boxed{\sigma_5 = \sigma_0}$

WHAT ARE THE ~~FIELDS~~ FIELDS?

$$E_I = 0, \quad E_{II} = 0, \quad E_{III} = \frac{\sigma_4 \hat{x}}{\epsilon_0} = -\frac{\sigma_5 \hat{x}}{\epsilon_0} = -\frac{\sigma_0 \hat{x}}{\epsilon_0}, \quad E_{IV} = 0.$$

FORCES: LEFT PLATE: $\sigma_1 = \sigma_2 = 0, \therefore \vec{F}_L = 0$
 MIDDLE PLATE:

$$\vec{F}_M = \frac{\sigma_4^2 \hat{x}}{2\epsilon_0} = \frac{\sigma_0^2 \hat{x}}{2\epsilon_0}$$

~~RIGHT PLATE:~~

RIGHT PLATE:

$$\vec{F}_R = -\frac{\sigma_5^2 \hat{x}}{2\epsilon_0} = -\frac{\sigma_0^2 \hat{x}}{2\epsilon_0}$$

NOTE THAT $\vec{F}_{NET} = \vec{F}_L + \vec{F}_M + \vec{F}_R = 0.$

Q: DID THAT HAVE TO BE TRUE?

A: YES! NET FORCE ON AN ISOLATED SYSTEM IS ALWAYS ZERO.

Q: WHAT IF THE "ELECTRON BEAM" WENT THROUGH A TINY HOLE IN THE MIDDLE PLATE AND LANDED ON THE LEFT PLATE INSTEAD?