

GRIFFITHS §3.3

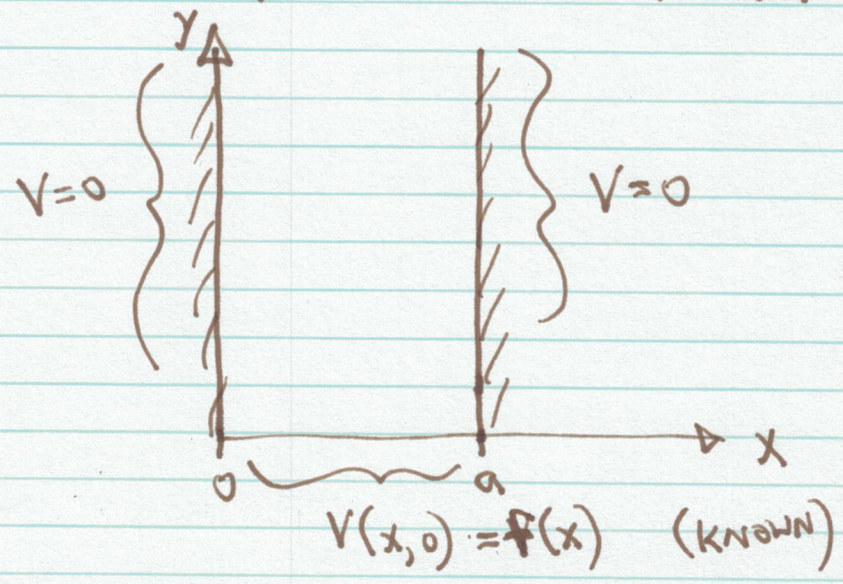
SEPARATION OF VARIABLES

WHEN WE HAVE A LINEAR PDE SUCH AS LAPLACE'S EQUATION,

$$\nabla^2 V = 0,$$

WITH SOME PARTICULAR BOUNDARY CONDITIONS, TYPICALLY WE CHOOSE COORDINATES THAT WORK WELL WITH THE B.C.'S AND THEN APPLY A CLASSIC TECHNIQUE THAT TRANSFORMS OUR PDE ON N INDEPENDENT VARIABLES INTO N ORDINARY DIFFERENTIAL EQUATIONS.

EXAMPLE $\nabla^2 V = 0$ IN A FINITE SLOT (2D)



$$\nabla^2 V = 0 : \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

ASSUME $V = X(x) Y(y)$. LAPLACE'S EQN. BECOMES:

$$X'' Y + X Y'' = 0$$

DIVIDE BY $V = XY$.

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

\nwarrow FUNCTION OF x ONLY \nearrow FUNCTION OF y ONLY

INFER THAT EACH TERM MUST EQUAL A CONSTANT (DISCUSS)

WE THEREFORE HAVE 2 ODES:

$$X'' = C_x X, \quad Y'' = C_y Y$$

AND REQUIRE $C_x + C_y = 0$.

Q: WHAT KINDS OF SOLUTIONS ARE ADMITTED BY

$$u'' = cu? \quad (\text{ASSUME } c \in \mathbb{R})$$

A: IF $c > 0$, $u'' = cu = k^2 u$, $k \in \mathbb{R}$

$$u = Ae^{kx} + Be^{-kx}$$

IF $c < 0$, $u'' = cu = -k^2 u$, $k \in \mathbb{R}$

$$u = A \cos kx + B \sin kx$$

Q: WHICH SORT OF SOLUTION MAKES SENSE FOR $X(y)$?

A: $X = e^{-ky}$ (DISCUSS!)

SO LET $C_y = +k^2$, $C_x = -k^2$

$$\Rightarrow X = A \cos kx + B \sin kx$$

BOUNDARY CONDITIONS: $V = 0$ @ $x = 0, a$

$$\Rightarrow A = 0, \quad k = \frac{\pi n}{a}, \quad n = 1, 2, 3, \dots$$

$$\therefore X = \sin\left(\frac{\pi n x}{a}\right)$$

Q: ARE WE DONE?

A: NO. NEED TO MATCH LOWER BOUNDARY ($y=0$).

GENERAL SOLUTION
$$V = \sum_{n=1}^{\infty} A_n \sin\left(\frac{\pi n x}{a}\right) e^{-\pi n y/a}$$

LOWER BOUNDARY:

$$* f(x) = V(x, 0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \quad [e^{-ny} \rightarrow 1 @ y=0]$$

HOW DO WE CHOOSE THE A_n ?

FIRST, REMEMBER THAT THE SINES FORM AN ORTHONORMAL BASIS:

$$\int_0^{\pi} \sin(m\theta) \sin(n\theta) d\theta = \begin{cases} \frac{\pi}{2}, & m=n \\ 0, & m \neq n \end{cases}$$

(PROVIDED $m, n \in \{1, 2, 3, \dots\}$)

AND ANY FUNCTION $g(\theta)$ ON $0 < \theta < \pi$ CAN BE WRITTEN AS A SUM OF THESE SINES *

$$\text{SIMILARLY, } \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \begin{cases} a/2, & m=n \\ 0, & m \neq n \end{cases}$$

NOW, HIT BOTH SIDES OF * WITH $\int_0^a dx \sin\left(\frac{m\pi x}{a}\right)$: **

$$\frac{a}{2} A_m = \int_0^a f(x) \sin\left(\frac{m\pi x}{a}\right) dx$$

$$\therefore V(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} \left[\int_0^a f(x') \sin\left(\frac{n\pi x'}{a}\right) dx' \right] \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a}$$

DISCUSS!

EXERCISE: DO $f(x) = V_0$ EXPLICITLY.

* YOU MIGHT WISH TO REVIEW FOURIER SERIES.
** FOURIER'S TRICK