

Griffiths § 3.3.2

①

SEPARATION OF VARIABLES IN SPHERICAL COORDINATES  
WITH AZIMUTHAL SYMMETRY ( $\partial/\partial\phi = 0$ )

$$\nabla^2 V = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

LET  $V = R(r) \Theta(\theta)$

$$\Theta \frac{\partial}{\partial r} (r^2 R') + \frac{R}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \Theta') = 0$$

DIVIDE BY  $R\Theta$ :

$$\frac{1}{R} \frac{\partial}{\partial r} (r^2 R') + \frac{1}{\Theta \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \Theta') = 0$$

FUNCTION OF  
 $r$  ONLY =  $C$

FUNCTION OF  $\theta$  ONLY =  $-C$

ODE FOR  $R$ :

$$2r R' + r^2 R'' = RC \quad *$$

TRY  $R = r^l$  :  $R' = l r^{l-1}$ ,  $R'' = l(l-1) r^{l-2}$

SO  $*$  BECOMES

$$2l r^l + l(l-1) r^l = r^l C \Rightarrow C = l(l+1).$$

IS THERE MORE THAN ONE  $l$  FOR A GIVEN  $C$ ? LET  $l \rightarrow l'$

$$l'^2 + l' - C = 0 \Rightarrow l' = \frac{-1 \pm \sqrt{1+4C}}{2} = \frac{-1 \pm \sqrt{1+4l^2+4l}}{2}$$

$$l' = \begin{cases} l \\ -(l+1) \end{cases}$$

$\therefore$  GENERAL SOLUTION TO  $\frac{1}{R} \frac{d}{dr} (r^2 R') = l(l+1)$

$$\text{IS } R = A_l r^l + B_l r^{-(l+1)}$$

Now our  $\Theta$  equation is

$$\frac{1}{\sin \Theta} \frac{d}{d\Theta} \left( \sin \Theta \frac{d(H)}{d\Theta} \right) + l(l+1) \Theta = 0 \quad **$$

MAKING THE SUBSTITUTION  $x = \cos \Theta$  (NOT CARTESIAN  $x$ ),

$$\frac{d}{d\Theta} = \frac{dx}{d\Theta} \frac{d}{dx}, \quad \text{AND } ** \text{ BECOMES}$$

$$\frac{d}{dx} \left[ (1-x^2) \frac{d(H)}{dx} \right] + l(l+1) H = 0, \quad -1 \leq x \leq 1$$

THIS IS LEGENRE'S EQUATION.

SOLUTIONS ARE LEGENRE POLYNOMIALS:

$$H(x) = P_l(x) \equiv \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2-1)^l, \quad -1 \leq x \leq 1$$

RODRIGUES FORMULA

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

THESE FORM AN ORTHOGONAL BASIS ON  $-1 \leq x \leq 1$

Q: WHAT IS THE DOMAIN IN  $\Theta$ , FOR  $x = \cos \Theta$ ?  
 A:  $0 \leq \Theta \leq \pi$

ORTHOGONALITY RELATION:

$$\int_{-1}^1 P_l(x) P_k(x) dx = \int_0^\pi P_l(\cos \Theta) P_k(\cos \Theta) \sin \Theta d\Theta$$

$$= \begin{cases} 0, & k \neq l \\ \frac{2}{2l+1}, & k = l \end{cases}$$

SO, IN SPHERICAL COORDINATES WITH  $\frac{\partial}{\partial \phi} = 0$ ,  
THE SOLUTION TO  $\nabla^2 V = 0$  IS:

$$V = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos \theta)$$

POTENTIAL

Q: WHAT IS THE <sup>POTENTIAL</sup> FIELD OF A UNIFORMLY CHARGED SPHERICAL SHELL, BOTH INSIDE & OUTSIDE?

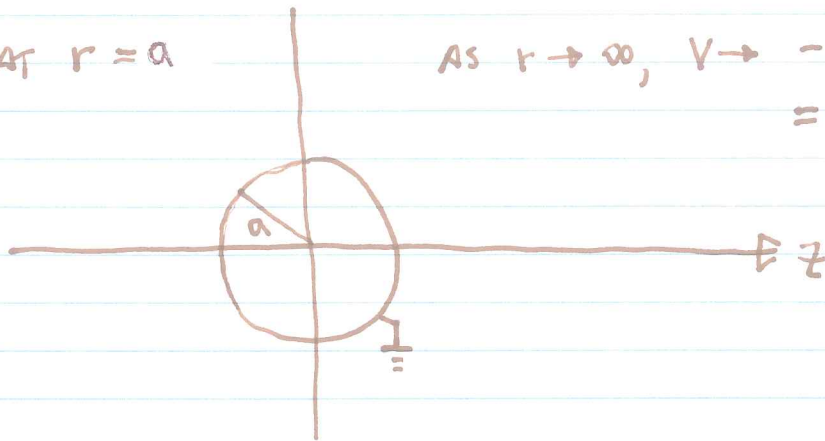
DISCUSS

EXAMPLE

CONSIDER THE CONDUCTING SPHERE, WITH EXTERNALLY APPLIED FIELD  $\vec{E}_\infty = E_0 \hat{z}$

$$V=0 \text{ AT } r=a$$

$$\begin{aligned} \text{AS } r \rightarrow \infty, V &\rightarrow -E_0 z \\ &= -E_0 r \cos \theta \end{aligned}$$



LOOKING BACK AT OUR GENERAL SOLUTION, NOTICE THAT

$$P_1(\cos \theta) = \cos \theta.$$

LOOK AT THE  $l=1$  TERM:

$$V_l = (A_l r + B_l r^{-2}) \cos \theta.$$

$$r=a: \quad 0 = A_l a + B_l a^{-2} \Rightarrow B_l = -a^3 A_l$$

$$r \rightarrow \infty: \quad -E_0 r \cos \theta = A_l r \cos \theta$$

CONCLUSION

$$V = (-E_0 r + a^3 E_0 r^{-2}) \cos \theta$$

Q: HOW WOULD THIS CHANGE IF WE PUT A NONZERO POTENTIAL ON THE SPHERE?