

Griffiths § 3.4 Review and Examples

Last time, we learned to approximate $V(\vec{r})$ far from a charge distribution $\rho(\vec{r}')$:



Monopole moment:

$$Q = \sum_i q_i = \int \rho d\tau; \quad V_{\text{MON}} = \frac{Q}{4\pi\epsilon_0 r}$$

Dipole moment:

$$\vec{p} = \sum_i \vec{r}_i q_i = \int \vec{r} \rho(\vec{r}) d\tau; \quad V_{\text{DIP}} = \frac{\vec{p} \cdot \vec{r}}{4\pi\epsilon_0 r^3}$$

Suppose we have a charge distribution $\rho(\vec{r})$ with nonzero Q and \vec{p} . Let's measure position from point \vec{a} instead of from the origin. Now $\vec{r} \rightarrow \vec{r} - \vec{a}$ in all our formulae:

$$\vec{p} \rightarrow \vec{p}_a = \int (\vec{r} - \vec{a}) \rho(\vec{r}) d\tau = \vec{p} - \vec{a} Q$$

Now, let's choose $\vec{a} = \vec{p}/Q$. Then $\vec{p}_a = \underline{\underline{0!}}$
 (This only works if $Q \neq 0$!)

Now $V = \frac{Q}{4\pi\epsilon_0 |\vec{r} - \vec{a}|} + \mathcal{O}(a^2)$
 \uparrow quadrupole & higher terms

THIS IS EQUIVALENT TO $V_{\text{MON}} + V_{\text{DIP}}$.

Examples of multipole expansion

$$\rho(\vec{r}) = \frac{k}{r^2} e^{-r^2/R^2} \cos \theta$$

Find the electric potential far from the origin.

By symmetry, we can see there is no net charge.

$$\vec{p} = \int \rho(\vec{r}') \vec{r}' d\tau'$$

In spherical coordinates, $\int d\tau'$

$$\vec{p} = \int_{r'=0}^{\infty} r'^2 dr' \int_{\phi'=0}^{2\pi} d\phi' \int_{\theta'=0}^{\pi} \sin \theta' d\theta' \left[\frac{k}{r'^2} e^{-r'^2/R^2} \cos \theta' \right] \vec{r}'$$

Q: What will go wrong here?

A: \hat{r}' depends on θ' and ϕ' !

We need constant unit vectors!

$$\vec{r}' = x' \hat{x} + y' \hat{y} + z' \hat{z}$$

But notice $\vec{p} = p \hat{z}$ by symmetry, so ignore \hat{x}, \hat{y} terms:

$$\vec{r}' \rightarrow z' \hat{z} = r' \cos \theta' \hat{z}$$

$$p = \int_{r'=0}^{\infty} k r' e^{-r'^2/R^2} dr' \int_{\phi'=0}^{2\pi} d\phi' \int_{\theta'=0}^{\pi} \cos^2 \theta' \sin \theta' d\theta'$$

$$= \frac{k R^2}{2} e^{-r'^2/R^2} \Big|_{r'=0}^{\infty} \left[-2\pi \frac{1}{3} \cos^3 \theta' \right]_{\theta'=0}^{\pi}$$

$$= \frac{k R^2}{2} \left[-\frac{2}{3} \pi (-1 - 1) \right]$$

$$= \frac{2}{3} \pi R^2 k \Rightarrow \vec{p} = \frac{2}{3} \pi R^2 k \hat{z}$$

Now, recall $V = \frac{\vec{p} \cdot \hat{r}}{4\pi \epsilon_0 r^2} = \frac{\vec{p} \cdot \hat{r}}{4\pi \epsilon_0 r^2} = \frac{pk \cos \theta}{4\pi \epsilon_0 r^2}$

$$\therefore \boxed{V = \frac{R^2 k \cos \theta}{6 \epsilon_0 r^2}}$$

What is the electric field due to a dipole of arbitrary orientation?

FROM Eq. 3.103,

$$\vec{E}_{\text{DIP}} = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos\theta \hat{r} + \sin\theta \hat{\theta}), \quad \vec{p} = p\hat{z}.$$

THE COORDINATE FREE FORM IS:

$$\vec{E}_{\text{DIP}} = \frac{1}{4\pi\epsilon_0 r^3} [3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}] \quad \leftarrow \text{TO PROVE. (Griffiths, eg. 3.104)}$$

NOTICE THAT $\cos\theta = \hat{z} \cdot \hat{r}$,

$$\hat{\theta} = -\hat{z} \sin\theta + \hat{\rho} \cos\theta,$$

$$\hat{r} = \hat{\rho} \sin\theta + \hat{z} \cos\theta$$

SO $\Rightarrow \hat{\rho} = \frac{1}{\sin\theta} (\hat{r} - \hat{z} \cos\theta).$

$$\vec{E}_{\text{DIP}} = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos\theta \hat{r} + \sin\theta [-\hat{z} \sin\theta + \hat{\rho} \cos\theta])$$

$$= \frac{p}{4\pi\epsilon_0 r^3} \left(2 \cos\theta \hat{r} + \sin\theta \left[-\hat{z} \sin\theta + \frac{\cos\theta}{\sin\theta} (\hat{r} - \hat{z} \cos\theta) \right] \right)$$

$$= \frac{p}{4\pi\epsilon_0 r^3} (3 \cos\theta \hat{r} - \hat{z} (\sin^2\theta + \cos^2\theta))$$

$$= \frac{1}{4\pi\epsilon_0 r^3} (3(\hat{z} \cdot \hat{r})\hat{r} - p\hat{z})$$

$$= \frac{1}{4\pi\epsilon_0 r^3} (3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}) \quad \checkmark$$