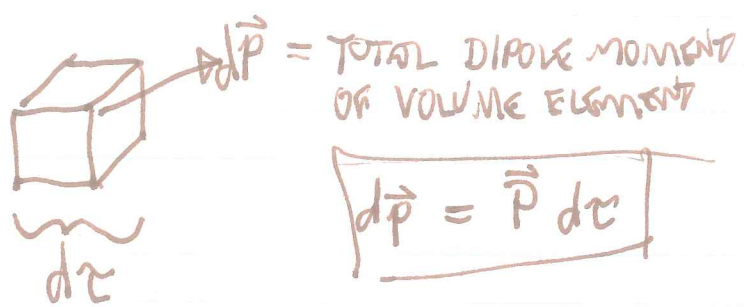


THE POLARIZATION FIELD, $\vec{P}(\vec{r})$ AND BOUND CHARGE

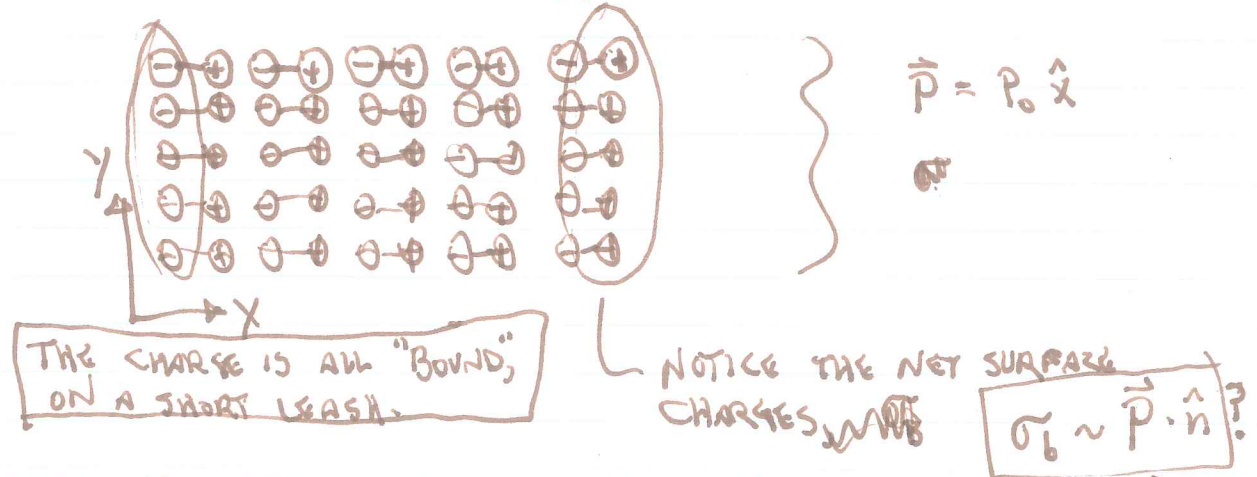
GRIFFITHS §4.2

WE NOW INTRODUCE A CONTINUOUS MEDIUM WITH A DIPOLE MOMENT PER VOLUME:



SO $\vec{P}(\vec{r})$ IS DIPOLE MOMENT PER VOLUME.

MICROSCOPICALLY, WE PICTURE THE MEDIUM AS A COLLECTION OF POLARIZED OBJECTS:



THERE ARE OTHER WAYS TO GET A NET DENSITY OF BOUND CHARGE:



★ NOTE HOW THE TWO TENTATIVE BOXED EXPRESSIONS ARE CONSISTENT WITH EACH OTHER! (DIVERGENCE THEOREM) ★

ARE THE UNITS CORRECT? YES!

ALTERNATIVE DERIVATION OF BOUND CHARGES (CCK VERSION)

IMAGINE A MATERIAL MADE OF "ATOMS" (OR MOLECULES) OF VOLUME V , AND CONTAINING EQUAL AND OPPOSITE CHARGES $+q$ (NUCLEI) AND $-q$ (ELECTRONS). IT MAY HELP TO THINK OF THESE CHARGES AS OVERLAPPING SPHERES.

IF THE NUCLEUS & ELECTRON CLOUD HAVE THE SAME CENTER, THE DIPOLE MOMENT IS ZERO. IF THE POSITIVE CHARGES ARE DISPLACED BY \vec{d} (OR THE NEGATIVE BY $-\vec{d}$), THEN $\vec{p} = q\vec{d}$. NOW THINK OF \vec{d} AS A ~~VECTOR~~ CONTINUOUS VECTOR FIELD OF DISPLACEMENTS. THE DIPOLE MOMENT PER VOLUME IS

$$\vec{p} = \frac{q\vec{d}}{V} = \rho \cdot \vec{d}$$

IF A LARGE VOLUME V CONTAINING MANY ATOMS HAS DISPLACEMENT $\vec{d}(\vec{r})$, THEN THE LOSS (OR GAIN) OF CHARGE THROUGH THE SURFACE IS

$$\Delta Q = \oint_{\sigma} (\rho \vec{d}) \cdot d\vec{a} = \int_V \nabla \cdot \vec{p} \, d\tau \quad (i)$$

THIS LOSS OF CHARGE CAN ALSO BE SEEN AS ARISING FROM NONZERO NET CHARGE DENSITY IN THE MATERIAL:

$$-\Delta Q = \int_V \rho_b \, d\tau \quad (ii)$$

COMPARING EQUATIONS (i) AND (ii), EVIDENTLY

$$\rho_b = -\nabla \cdot \vec{p} \quad (iii)$$

IF THE MATERIAL HAS AN EDGE, THEN APPLICATION OF THE DIVERGENCE THEOREM ^{TO (iii)} TELLS US THAT THERE WILL BE A SURFACE CHARGE

$$\sigma_b = \vec{p} \cdot \hat{n}$$

(2)

WORTHY APPROACH:

$$V(\vec{r}) = \int \frac{\hat{n} \cdot \vec{P}(\vec{r}') d\tau'}{4\pi\epsilon_0 r^2}$$

THIS IS HOW
GRIFFITHS
DOES IT.

BUT $\frac{\hat{n}}{r^2} = \nabla' \left(\frac{1}{r} \right)$, so

$$V = \frac{1}{4\pi\epsilon_0} \int \vec{P}(\vec{r}') \cdot \nabla' \left(\frac{1}{r} \right) d\tau'$$

NOW INTEGRATE BY PARTS USING THE PRODUCT RULE:

$$\nabla \cdot \left(\frac{\vec{P}}{r} \right) = \vec{P} \cdot \nabla' \left(\frac{1}{r} \right) + \frac{1}{r} \nabla' \cdot \vec{P}$$

$$\Rightarrow V = \frac{1}{4\pi\epsilon_0} \int \left[\nabla' \cdot \frac{\vec{P}}{r} - \frac{1}{r} \nabla' \cdot \vec{P} \right] d\tau'$$

$$\therefore V = \oint \frac{\vec{P} \cdot \hat{n} da}{4\pi\epsilon_0 r} - \int \frac{\nabla' \cdot \vec{P}(\vec{r}')}{4\pi\epsilon_0 r} d\tau'$$

OUR TENTATIVE IDENTIFICATIONS WERE:

$$\begin{array}{l} \rho_b = -\nabla \cdot \vec{P} \\ \sigma_b = \vec{P} \cdot \hat{n} \end{array}$$

ARE THESE CONFIRMED BY OUR INTEGRAL FOR $V(\vec{r})$?

$$V(\vec{r}) = \oint \frac{\sigma_b da}{4\pi\epsilon_0 r} + \int \frac{\rho_b(\vec{r}') d\tau'}{4\pi\epsilon_0 r}$$

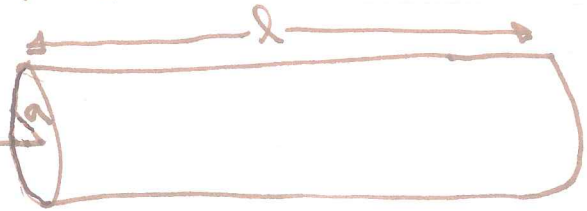
AHA!

EXAMPLE:

FIND THE CHARGE DENSITY AND ELECTRIC FIELD IN A LONG, NARROW CYLINDER WITH

$$\vec{P} = P_0 \hat{s}$$

$z \leftarrow$



$$\rho_b = -\nabla \cdot \vec{P} = -P_0 \nabla \cdot \hat{s} = \boxed{\frac{-P_0}{s}}$$

Q: IS THERE A LINE CHARGE (λ AT THE AXIS?)

$$A: \int_V \rho_b dV = \lambda l = -\int_V \vec{P} \cdot d\vec{\tau} = -P_0 2\pi s l \quad \leftarrow \text{GAUSS'S THEOREM NOT GAUSS'S LAW!}$$

$$\therefore \lambda = -2\pi s P_0 \quad \text{--- ENCLOSED CHARGE PER LENGTH}$$

AS $s \rightarrow 0$, NO ENCLOSED CHARGE.

SURFACE CHARGE AT $s = a$:

$$\boxed{\sigma_b = P_0 \hat{s} \cdot \hat{n} = P_0}$$

THE ELECTRIC FIELD JUST COMES FROM THE BOUND CHARGES:

$$\oint \vec{E} \cdot d\vec{a} = Q_{enc}/\epsilon_0$$

$$\Rightarrow E \cdot 2\pi s l = \lambda l / \epsilon_0 = -2\pi s P_0 / \epsilon_0$$

$$\Rightarrow \boxed{\vec{E} = \frac{-P_0 \hat{s}}{\epsilon_0} = \frac{-\vec{P}}{\epsilon_0}}$$

Q: WHAT IS THE EXTERNAL FIELD?

A: \emptyset . DISCUSS!

\rightarrow YOU WILL PROVE ~~IT~~ IN GENERAL THE CHARGE NEUTRALITY OF A POLARIZED OBJECT (PROB. 4.14).

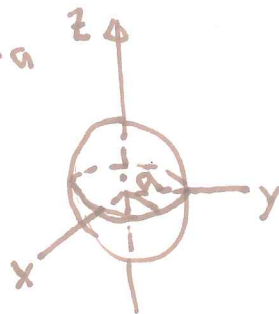
REVIEW — CONDUCTING SPHERE IN A UNIFORM FIELD, $E_0 \hat{z}$.

WE FOUND $V = (-E_0 r + a^3 E_0 r^{-2}) \cos \theta$, $r > a$
(AND $V=0$, $r < a$)

THE RESULTING ELECTRIC FIELD IS:

$$\vec{E} = -\nabla V$$

$$\vec{E}(r) = \begin{cases} \hat{r} (E_0 + 2a^3 E_0 r^{-3}) \cos \theta \\ + \hat{\theta} (-E_0 + a^3 E_0 r^{-3}) \sin \theta, & r > a \\ 0, & r < a \end{cases}$$



BY GAUSS'S LAW, THE SURFACE CHARGE DENSITY IS

$$\sigma = \epsilon_0 E_r (r=a) = 3E_0 \epsilon_0 \cos \theta.$$

~~BEARING THIS SOLUTION~~

WHAT IS THE FIELD DUE TO σ ? FOR THAT WE
NEGLECT THE FIRST TERM $\vec{E} = E_0 [\hat{r} \cos \theta - \hat{\theta} \sin \theta] = E_0 \hat{z}$,
WHICH IS THE EXTERNAL FIELD.

SUBTRACTING THIS OFF FOR BOTH $r < a$ AND $r > a$,

$$\vec{E}_\sigma = \begin{cases} 2E_0 a^3 r^{-3} \cos \theta \hat{r} + E_0 a^3 r^{-3} \sin \theta \hat{\theta}, & r > a \\ -E_0 \hat{z}, & r < a \end{cases}$$

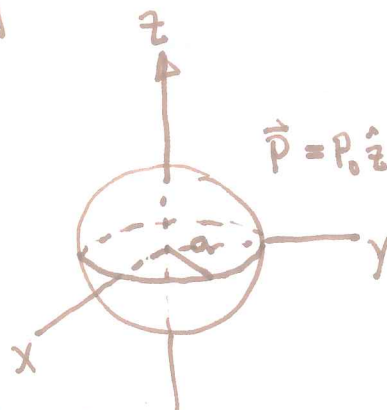
THIS IS WHAT CANCELLED THE
EXT. FIELD, SO THAT $\vec{E} = 0$
IN THE CONDUCTOR.

EXAMPLE

SPHERE WITH UNIFORM POLARIZATION
(FIND CHARGE & FIELDS)

THIS GIVES RISE TO A SURFACE CHARGE:

$$\sigma_b = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{r} = \boxed{P_0 \cos \theta}$$



DOES THIS LOOK FAMILIAR?

IT IS OUR OLD FRIEND. WE KNOW
ALREADY THAT THE FIELD IS:

$$\vec{E} = \frac{P_0}{3\epsilon_0 \epsilon_0} \vec{E}_r,$$

$$\vec{E} = \begin{cases} \frac{4}{3} \frac{P_0 a^3}{\epsilon_0 r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}), & r > a; \\ -\frac{P_0 \hat{z}}{3\epsilon_0}, & r < a. \end{cases}$$

Q: WHAT IS THE TOTAL DIPOLE MOMENT OF THE SPHERE?

$$A: \vec{p} = \int \vec{P} d\tau = \left(\frac{4}{3} \pi a^3\right) (P_0 \hat{z}) = \frac{4}{3} \pi a^3 P_0 \hat{z}$$

Q: IN THE DIPOLE APPROXIMATION, WHAT IS THE FIELD?

A: SAME AS THE $r > a$ SOLUTION ABOVE! (NOT ALL GEOMETRIES WOULD GIVE A PURE DIPOLE AS THE SPHERE DOES!)

Exercise: Sketch the field of a bar electret (Griffiths prob. 4.11, part iii)