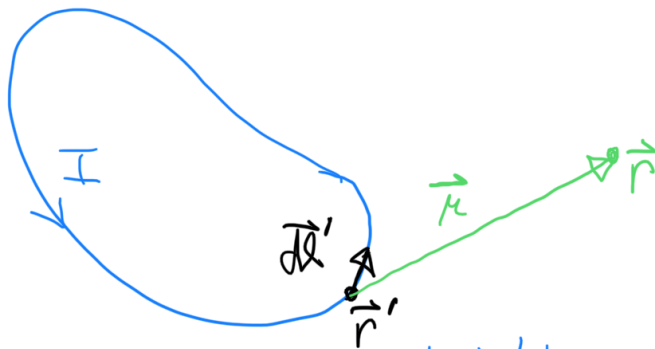


## § 5.2 Biot-Savart Law

Jean-Baptiste Biot and Félix Savart (1820)

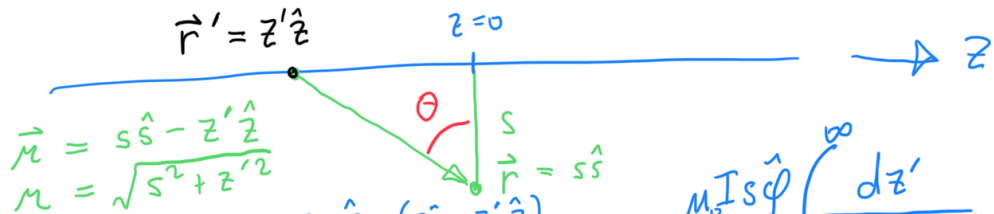
$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell}' \times \hat{r}}{r^2} \quad \left. \vphantom{\int} \right\} \begin{array}{l} \text{The Biot-Savart Law is for} \\ \text{magnetostatics what Coulomb's} \\ \text{Law is for electrostatics.} \end{array}$$



Example: Infinite, straight wire

$$d\vec{\ell}' = \hat{z} dz'$$

$$\vec{r}' = z' \hat{z}$$



$$\vec{r} = s\hat{s} - z'\hat{z}$$

$$r = \sqrt{s^2 + z'^2}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\hat{z} \times (s\hat{s} - z'\hat{z})}{(s^2 + z'^2)^{3/2}} dz' = \frac{\mu_0 I s \hat{\phi}}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(s^2 + z'^2)^{3/2}}$$

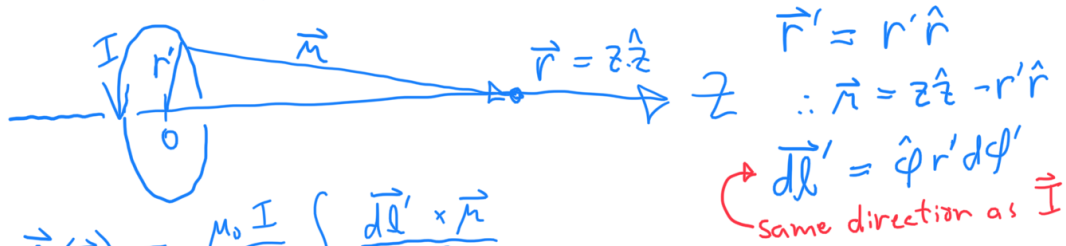
$$= \frac{\mu_0 I \hat{\phi}}{4\pi s^2} \int_{-\infty}^{\infty} \frac{dz'}{\left[1 + \left(\frac{z'}{s}\right)^2\right]^{3/2}}$$

$$\frac{z'}{s} = \tan \theta = \frac{\sin \theta}{\cos \theta} \Rightarrow \left[1 + \left(\frac{z'}{s}\right)^2\right]^{3/2} = \left(\frac{1}{\cos^2 \theta}\right)^{3/2} = \frac{1}{\cos^3 \theta}$$

$$dz' = s \left(1 + \frac{\sin^2 \theta}{\cos^2 \theta}\right) d\theta = \frac{-s d\theta}{\cos^2 \theta}$$

$$\vec{B} = \frac{\mu_0 I \hat{\phi}}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \frac{\mu_0 I \hat{\phi}}{2\pi s}$$

B.S. Example: Find  $\vec{B}$  on axis...



$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{r}}{r^3}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\hat{\phi} \times (z \hat{z} - r' \hat{r})}{(z^2 + r'^2)^{3/2}} r' d\phi' = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{z \hat{r} - r' \hat{z}}{(z^2 + r'^2)^{3/2}} r' d\phi'$$

integrates to zero

What's left of the integrand is a constant!

$$\int d\phi \rightarrow 2\pi$$

$$\vec{B} = \frac{\mu_0 I}{2} \frac{r'^2}{(z^2 + r'^2)^{3/2}} \hat{z}$$

Q: What is  $\int_{-\infty}^{\infty} \vec{B} \cdot \hat{z} dz$ ?

A:  $\mu_0 I$  ... Does that look familiar?

Discuss:

- How hard would it be to calculate  $\vec{B}$  in the xy-plane?
- How would you find  $\vec{B}$  on axis for the charged spinning platter from our last lecture?

## Generalizations of Biot-Savart Law

Let's rewrite with  $\vec{I}(\vec{r}')$  as a vector:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I}(\vec{r}') \times \vec{r}}{r^3} dl'$$

Now consider surface & volume currents:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \vec{r}}{r^3} d\tau'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r}') \times \vec{r}}{r^3} d\tau'$$

This is the most general form, since line & surface currents can be expressed as volume currents (using delta functions).

What is  $\nabla \cdot \vec{B}$ ?

$$\nabla \cdot \int \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau' = ?$$

$\nabla$  operates on  $\vec{r}$ , not  $\vec{r}'$ . Move inside!

Identity:

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\begin{aligned} \nabla \cdot \left( \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) &= 0 - \vec{J}(\vec{r}') \cdot \nabla \times \left[ \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right] \\ &= 0. \quad (\text{In general, } \nabla \times (f(r) \hat{r}) = 0.) \end{aligned}$$