

Ampere's Law § 5.3

What is $\nabla \times \vec{B}$?

$$\nabla \times \vec{B} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') \times \hat{r}}{r^3} d\tau'$$

Just like with $\nabla \cdot \vec{B}$, our $\nabla \times$ operator will move inside and hit just $\hat{r} = \frac{\vec{r} - \vec{r}'}{r}$.

Identity:

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \left[-(\vec{J}(\vec{r}') \cdot \nabla) \frac{\hat{r}}{r^3} + \vec{J} \left(\nabla \cdot \frac{\hat{r}}{r^3} \right) \right] d\tau'$$

Let's look at this term. (why?)

$$-(\vec{J}(\vec{r}') \cdot \nabla) \frac{\hat{r}}{r^2} = +(\vec{J}(\vec{r}') \cdot \nabla') \frac{\hat{r}}{r^2}$$

Consider just x-component:

$$(\vec{J}(\vec{r}') \cdot \nabla') \frac{x-x'}{r^3} = \nabla' \cdot \left[\vec{J}(\vec{r}') \frac{x-x'}{r^3} \right] - \frac{x-x'}{r^3} [\nabla' \cdot \vec{J}]$$

{ Identity: $\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + (\vec{A} \cdot \nabla) f$ }

Now, $\nabla' \cdot \vec{J}(\vec{r}') = \frac{\partial \rho}{\partial t} = 0$ (magnetostatics)

$$\Rightarrow \int (\vec{J}(\vec{r}') \cdot \nabla') \frac{x-x'}{r^3} d\tau' = \oint \left[\vec{J}(\vec{r}') \frac{x-x'}{r^3} \right] \cdot d\vec{A} = 0$$

(If we are integrating over all space, or at least a volume large enough to contain all the currents). So,

$$\int (\vec{J}(\vec{r}') \cdot \nabla) \frac{\hat{r}}{r^2} d\tau' = 0.$$

$4\pi \delta^3(\vec{r} - \vec{r}')$
(chapter 1)

We are left with

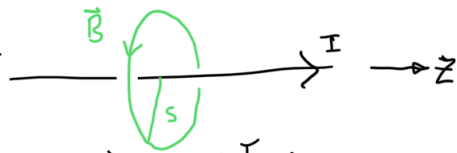
$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}') \left[\nabla \cdot \frac{\hat{r}}{r^3} \right] d\tau' = \mu_0 \int \vec{J}(\vec{r}') \delta^3(\vec{r} - \vec{r}') d\tau'$$

$$\therefore \nabla \times \vec{B} = \mu_0 \vec{J}; \quad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \vec{J} \cdot d\vec{A} = \mu_0 I_{enc.}$$

AMPERE'S LAW

Ampere's Law Examples:

① Long, straight wire



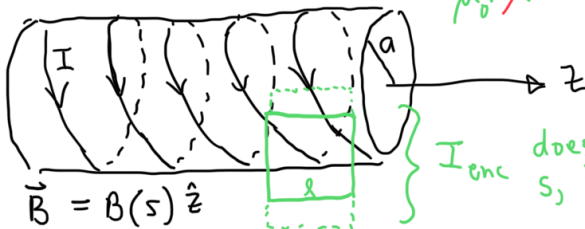
$$\mu_0 I = \oint \vec{B} \cdot d\vec{\ell} = 2\pi s B \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

② Solenoid: N turns, length $L \gg a$

If it is very tightly wound,

$$\vec{K} = \frac{NI}{L} \hat{\phi}$$

By symmetry, $\vec{B} = B(s) \hat{z}$



Ampere: $\mu_0 K l = (B_{in} - B_{out}) l$

I_{enc} does not depend on s , inside or out!

Ampere's Law $\Rightarrow B(s) = \begin{cases} B_{in}, & s < a \\ B_{out}, & s > a \end{cases}$

Moreover, considering $s \rightarrow \infty$, $B_{out} = 0$, so $B_{in} = \mu_0 K$

$$\vec{B} = \begin{cases} \mu_0 K \hat{z} = \frac{\mu_0 NI}{L} \hat{z}, & s < a \\ 0, & s > a \end{cases}$$

- Note that the result doesn't depend on a !
- We assumed a circular cross-section, but that doesn't matter either, so long as the cross-sectional shape & size does not depend on z .

DISCUSS