

§5.4. Vector Potential

I lectured directly from Griffiths, but gave the following unique example.

TRACE INSTRUMENT MECHANICAL LAYOUT
FULL SECTION EXCEPT AS NOTED

$\vec{B} = B\hat{\phi}$
($B < 0$)

$d\vec{l} = -\hat{\phi} dl$

Find: \vec{B}, \vec{J}

$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$
 $2\pi s B = -\mu_0 V/R$ ($\vec{I} = -I\hat{z}$)

$$\vec{B} = \frac{-\mu_0 V}{2\pi s R} \hat{\phi}, z > 0$$

$\vec{J} = \frac{V}{R} \frac{\hat{r}}{2\pi r^2} = \frac{V}{R} \frac{\hat{z} \cos\theta + \hat{s} \sin\theta}{2\pi (s^2 + z^2)}$

half-space $z \leq 0$

$\oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J}(\vec{r}') \cdot d\vec{a}'$

$\vec{B} = B\hat{\phi}$

$2\pi s B = \mu_0 \frac{V}{R} \int \frac{\cos\theta}{2\pi (s'^2 + z^2)} s' ds' d\phi$ ($d\phi = 2\pi$)

$= \mu_0 \frac{V}{R} z \int_0^s \frac{s' ds'}{(s'^2 + z^2)^{3/2}} = \mu_0 \frac{V}{R} \left[\frac{-1}{\sqrt{s'^2 + z^2}} \right]_{s'=0}^s$

$= \mu_0 \frac{V}{R} \left[\frac{-1}{\sqrt{s^2 + z^2}} + \frac{1}{|z|} \right] = \mu_0 \frac{V}{R} \left[-\cos\theta - \frac{z}{|z|} \right]$ $z < 0$

$\vec{B} = \mu_0 \frac{V}{R} \frac{\hat{\phi}}{2\pi s} [-1 - \cos\theta]$

Note how the solution joins together smoothly at $z=0$.

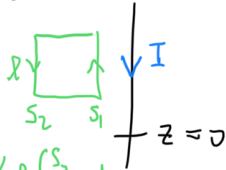
Now, beginning from

$$\vec{B} = \begin{cases} -\mu_0 \frac{V}{R} \frac{\hat{\phi}}{2\pi s}, & z > 0, \\ -\mu_0 \frac{V}{R} \frac{\hat{\phi}}{2\pi s} (1 + \cos \theta), & z \leq 0, \end{cases} \quad \cos \theta = \frac{z}{\sqrt{s^2 + z^2}}$$

What is the vector potential, \vec{A} , $\Rightarrow \vec{B} = \nabla \times \vec{A}$

and $\nabla \cdot \vec{A} = 0$? Let's consider $z > 0$ first.

It is often useful to begin by guessing \vec{A} is directed parallel to the current:



$\vec{A} = A(s) \hat{z}$. Now,

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} \quad \rightarrow \quad [A(s_1) - A(s_2)] = -\mu_0 \frac{V}{R} l \int_{s_1}^{s_2} \frac{ds}{2\pi s}$$

$$\Rightarrow A(s_1) - A(s_2) = \mu_0 \frac{V}{R 2\pi} [\ln(s_1) - \ln(s_2)]$$

We can't conveniently set $A(0) = 0$ or $A(\infty) = 0$.

But we can simply choose

$$A(s) = \mu_0 \frac{V}{R} \frac{\ln s}{2\pi} \quad \left. \begin{array}{l} A \rightarrow -\infty \text{ as } s \rightarrow 0; \\ A \rightarrow \infty \text{ as } s \rightarrow \infty. \end{array} \right\}$$

For $z < 0$, let's try $\vec{A} = -\mu_0 \frac{V}{R 2\pi} [f(s) \hat{z} + g(z) \hat{s}]$

$$(\nabla \times [\vec{A}])_{\phi} = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial s} \right) = \frac{1 + \cos \theta}{s} = \frac{1}{s} + \frac{z}{s \sqrt{s^2 + z^2}}$$

This looks tricky! How about $\vec{A} = -\mu_0 \frac{V}{R 2\pi} [f(r) \hat{\theta} + g(\theta) \hat{r}]$

$$(\nabla \times [\vec{A}])_{\phi} = \frac{1}{r} \left(\frac{\partial}{\partial r} [r f] - \frac{\partial g}{\partial \theta} \right) = \frac{1 + \cos \theta}{s} = \frac{1 + \cos \theta}{r \sin \theta}$$

Aha! $f = 0, \quad g = -\int d\theta \frac{1 + \cos \theta}{\sin \theta} = -\ln(1 - \cos \theta)$ (ha!)

$$\vec{A} = \begin{cases} -\mu_0 \frac{V}{R} \frac{\hat{r}}{2\pi} \ln(1 - \cos \theta), & z < 0 \\ \mu_0 \frac{V}{R} \frac{\ln s}{2\pi} \hat{z}, & z > 0 \end{cases}$$

Unfortunately, this solution is discontinuous at $z=0$, even though \vec{B} is not.

I could have chosen $f = \frac{1 + \cos \theta}{\sin \theta}, \quad g = 0$.
This would also be discontinuous with the $z > 0$ solution, and it would have $\nabla \cdot \vec{A} \neq 0$.

As you can see, it's not easy for this problem to work out a vector potential that is well-behaved across $z=0$.