

Physics 567

Homework 5

Advection of Atmospheric Aerosols

Due Friday, April 26, 2024

Convective atmospheric eddies (thermally driven vortices with the axis parallel to the ground) of 1 km size often form just above the Earth's surface. These eddies pick up dust particles and water droplets (aerosols) and carry (advect) them to overlying mesoscale (10 km) eddies that reach up to the lower boundary of the stratosphere (10-20 km altitude). In this assignment, you will construct an elementary 2D simulation of the trajectory of an aerosol particle (trapped?) in an idealized system of steady, 1 km convective eddies.¹

The aerosol particle experiences two forces: gravity and aerodynamic drag. The equation of motion is

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{y}} + \frac{1}{2m}C_D A\rho|\mathbf{w} - \dot{\mathbf{x}}|(\mathbf{w} - \dot{\mathbf{x}})$$

where $\mathbf{x} = (x, y)$ is the particle position. The y -axis points up. The dots represent time derivatives, A is the cross-sectional area of the (spherical) particle, g is the acceleration of gravity, C_D is the drag coefficient, ρ is the density of the atmosphere, and m is the mass of the particle. Assume that the particle has the density of water. The coefficient of drag for a sphere is highly dependent on

¹The real atmosphere has an evolving, complex chain of eddy systems, including smaller scale (100 m) turbulent eddies that serve to enhance particle diffusion. Aerosols also experience diffusion due to Brownian motion. We will ignore these effects.

the Reynolds Number (Re):

$$C_D \approx \frac{24}{Re} \left(1 + \frac{Re^{2/3}}{6} \right), \quad Re \lesssim 1000,$$

$$\text{where} \quad Re \equiv \frac{\rho v d}{\eta}.$$

The speed v is the magnitude of the velocity of the fluid relative to the particle ($v \equiv |\mathbf{w} - \dot{\mathbf{x}}|$); d is the particle diameter, and η is the viscosity of air.

We need a plausible expression for the convective velocity field $\mathbf{w}(x, y)$. In fluid flow, conservation of mass is embodied in the *continuity equation*,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{w}).$$

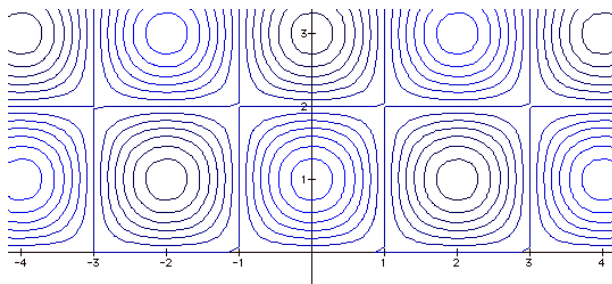
In our simplified world, fluid flow will be steady ($\frac{\partial}{\partial t} = 0$) and incompressible ($\rho = \text{constant}$). The velocity field therefore has zero divergence. This is easily accomplished in 2D by defining \mathbf{w} in terms of a stream function $\psi(x, y)$:

$$w_x = -\frac{\partial \psi}{\partial y}, \quad w_y = \frac{\partial \psi}{\partial x}.$$

We choose the stream function

$$\psi(x, y) = aw_0 \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi y}{2a}\right).$$

The contours of the stream function are the streamlines (field lines of the velocity field):



Note that the vortex directions alternate in a checkerboard pattern. This is (crudely) what convection cells look like in cross-section. Note that the period of circulation for a given air molecule depends on which streamline it is on. With that caveat, we somewhat arbitrarily define the convective turnover time as

$$\tau \equiv \frac{2\pi a}{w_0}.$$

Your assignment:

1. Run simulations for particle diameters $d = 1, 10, 100, 1000 \mu\text{m}$ (microns). For each aerosol particle, produce a plot of the trajectory x, y for 10 convective turnover times. Use the following initial conditions:

$$\begin{aligned} x_0 &= 0, & y_0 &= 1.5 \text{ km}, \\ \dot{x}_0 &= 0, & \dot{y}_0 &= 0. \end{aligned}$$

2. Find the largest size of particle that can be trapped in the vortex with the above initial conditions. We will deem the particle trapped if it does not reach ground ($y = 0$) within 10 convective turnover times.

Use the following parameter values in your simulation:

particle density	...	1000 kg/m ³
air density	ρ	1 kg/m ³
vortex speed	w_0	10 m/s
vortex radius	a	1000 m
air viscosity	η	$2 \times 10^{-5} \text{ N s/m}^2$
gravitational acceleration	g	10 m/s ²

Turn in the following by email to the grader:

1. Well-commented source code.
2. Trajectory plots for each of the particle sizes specified, for 10 turnover times.
3. Answer to the question: what is the largest size particle that will remain trapped? Your answer should be within $0.1 \mu\text{m}$ or better. Include program output to substantiate your answer.

The last part will be more tractable if you automate your search. Please be advised that whether a particle is trapped may not be a monotonic function of particle size.